Supplementary Appendix for “Asymptotic Properties of Empirical Quantile-Based Estimators”

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Abstract

This appendix contains additional lemmas used to prove Theorems 1 and 2.

S.1 Lemmas for Theorem 1

Lemma S.1.1 *For all* $j \in \{1, 2, 3, 4\}$*:* $R_j = o_P(1)$ *as* $N \to \infty$*.*

Proof: First, we show that $R_1 = o_P(1)$. By the triangle inequality, we have

$$\begin{aligned}|R_1| &\leq \sqrt{N}\left|F_Y^{-1}(\xi_{(n_1)})\right|\left|F_U(1) - F_U(\xi_{(n_1)})\right| + \sqrt{N}\left|F_Y^{-1}(\xi_{(1)})\right|\left|F_U(1/n_1) - F_U(\xi_{(1)})\right| \\ &= \sqrt{N}\left|Y_{(n_1)}\right|\left|1 - F_U(\xi_{(n_1)})\right| + \sqrt{N}\left|Y_{(1)}\right|\left|F_U(1/n_1) - F_U(\xi_{(1)})\right|.\end{aligned} \tag{S.1.1}$$

Lemma 4 and Markov’s inequality imply

$$\sqrt{N}\left|Y_{(1)}\right| = O_P\left(n_1^{\frac{1}{2}+d_1}\right) \quad \text{and} \quad \sqrt{N}\left|Y_{(n_1)}\right| = O_P\left(n_1^{\frac{1}{2}+d_2}\right),$$

and it suffices to show that

$$\left|F_U(1/n_1) - F_U(\xi_{(1)})\right| = o_P\left(n_1^{-\frac{1}{2}-d_1}\right) \quad \text{and} \quad \left|1 - F_U(\xi_{(n_1)})\right| = o_P\left(n_1^{-\frac{1}{2}-d_2}\right).$$

We show the second result only (the first can de obtained analogously). By the triangle inequality and the mean value theorem, there exists $c_{n_1} \in ((n_1-1)/n_1 \wedge \varepsilon_{(n_1)}, (n_1-1)/n_1 \vee \varepsilon_{(n_1)})$ such that

$$\begin{aligned}\left|1 - F_U(\xi_{(n_1)})\right| &\leq \left|1 - F_U((n_1-1)/n_1)\right| + \left|F_U((n_1-1)/n_1) - F_U(\xi_{(n_1)})\right| \\ &= \int_{\frac{n_1-1}{n_1}}^{1} f_U(u)du + f_U(c_{n_1})\left(1 - \xi_{(n_1)} + \frac{1}{n_1}\right) \\ &\lesssim \int_{\frac{n_1-1}{n_1}}^{1} u^{-b_1}(1-u)^{-b_2}du + c_{n_1}^{-b_1}(1-c_{n_1})^{-b_2}\left(1 - \xi_{(n_1)} + \frac{1}{n_1}\right)\end{aligned}$$

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$$
\begin{aligned}
&\lesssim \int_0^{\frac{1}{n_1}} v^{-b_2} dv + \left(2^{b_1} + \xi_{(n_1)}^{-b_1}\right)\left(n_1^{b_2} + (1-\xi_{(n_1)})^{-b_2}\right)\left(1 - \xi_{(n_1)} + \frac{1}{n_1}\right) \\
&\lesssim n_1^{b_2-1} + n_1^{b_2-1}\left(2^{b_1} + \xi_{(n_1)}^{-b_1}\right)\left(1 + [n_1(1-\xi_{(n_1)})]^{-b_2}\right)\left(n_1(1-\xi_{(n_1)}) + 1\right) \\
&= n_1^{b_2-1}\left(1 + O_P(1)\right),
\end{aligned}
$$

where the last line follows from $\xi_{(n_1)} \xrightarrow{p} 1$, $n_1(1-\xi_{(n_1)}) \xrightarrow{d}$Exponential(1), and the continuous mapping theorem. The result follows from $b_2 + d_2 < 1/2$ by Assumption 2(iv).

Second, we show that $R_2 = o_P(1)$. By Assumption 2(ii)–(iii),

$$
\begin{aligned}
\sqrt{N}\left|\int_0^{\xi_{(1)}} F_Y^{-1} dF_U\right| &\lesssim \sqrt{n_1}\mathbb{1}\left\{\xi_{(1)} \geq 1/2\right\}\left|\int_0^1 F_Y^{-1} dF_U\right| + \sqrt{n_1}\mathbb{1}\left\{\xi_{(1)} < 1/2\right\}\int_0^{\xi_{(1)}} t^{-b_1-d_1} dt \\
&\lesssim \sqrt{n_1}\mathbb{1}\left\{\xi_{(1)} \geq 1/2\right\} + n_1^{b_1+d_1-1/2}\left[n_1\xi_{(1)}\right]^{1-b_1-d_1}.
\end{aligned}
$$

The first term is $o_P(1)$ since it converges to zero in L_1:

$$
E\left[\sqrt{n_1}\mathbb{1}\left\{\xi_{(1)} \geq 1/2\right\}\right] = \sqrt{n_1}\,\mathrm{Pr}\left(\xi_{(1)} \geq 1/2\right) = \frac{\sqrt{n_1}}{2^{n_1}} = o(1).
$$

That the second term is $o_P(1)$ follows from $E[\xi_{(1)}] = 1/(n_1+1)$ and Jensen's inequality since $b_1 + d_1 < 1/2$. This shows that

$$
\sqrt{N}\left|\int_0^{\xi_{(1)}} F_Y^{-1} dF_U\right| = o_P(1). \tag{S.1.2}
$$

An analogous reasoning shows that

$$
\sqrt{N}\left|\int_{\xi_{(n_1)}}^1 F_Y^{-1} dF_U\right| = o_P(1). \tag{S.1.3}
$$

It follows that $R_2 = o_P(1)$.

Third, we show that $R_3 = o_P(1)$. By the mean value theorem, there exists $u_{n_1}(t) \in (\mathbb{G}_{n_1}(t), t)$ such that

$$
R_3 = \sqrt{N}\int_{\xi_{(1)}}^{\xi_{(n_1)}} \underbrace{\left[f_U - f_U \circ u_{n_1}\right]}_{=:\nu_{n_1}}\left[\mathbb{G}_{n_1} - \mathrm{I}\right] dF_Y^{-1}.
$$

By Assumption 2(iv), there exists $\delta > 0$ such that $b_j + d_j < 1/2 - \delta$. Further, let $\delta_j > 0$ be such that $\delta > 2(\delta_1 \vee \delta_2)$ and

$$
b_j + d_j < 1/2 - \delta - \delta_j.
$$

Then, let $q(t) = t^{1/2-\delta_1}(1-t)^{1/2-\delta_2}$. From what precedes, we have

$$
|R_3| \leq \sup_{t\in(0,1)}\left|\frac{\sqrt{n_1}(\mathbb{G}_{n_1}(t) - t)}{q(t)}\right|\int_{\xi_{(1)}}^{\xi_{(n_1)}} |\nu_{n_1}(t)| q(t) dF_Y^{-1}(t). \tag{S.1.4}
$$

We now show that the integral term in (S.1.4) tends to 0 in L_1. Let $f_{n_1}(t) := \mathbb{1}\left\{\xi_{(1)} < t < \xi_{(n_1)}\right\}|\nu_{n_1}(t)|$ and let μ denote the measure defined as the product of the probability measure on Ω, the probability space on which all random variables are defined, with the measure ρ on $(0,1)$ such that $d\rho/dF_Y^{-1} = q$. By Lemma 6, $\mu(\Omega \times (0,1)) = \int_0^1 q(t) dF_Y^{-1}(t) < \infty$. Moreover,

$$
E\left[\int_{\xi_{(1)}}^{\xi_{(n_1)}} |\nu_{n_1}(t)| q(t) dF_Y^{-1}(t)\right] = \int_{\Omega\times(0,1)} f_{n_1} d\mu.
$$

First, we show that f_{n_1} tends to 0 μ-almost surely. By almost-sure convergence of $\mathbb{G}_{n_1}(t)$ to t, we have, for all $t \in (0,1)$, $u_{n_1}(t) \xrightarrow{\text{a.s.}} t$. Then, by continuity of f_U, $\nu_{n_1}(t) \xrightarrow{\text{a.s.}} 0$ for all $t \in (0,1)$. This shows that $f_{n_1} \to 0$ μ-almost-surely. Second, by Corollary(i)–(ii) of Theorem 16.14 in (Billingsley, 1995, p. 218), it is sufficient to show that f_{n_1} is uniformly integrable, i.e., that

$$\lim_{M\to\infty} \limsup_{N\to\infty} \int \mathbb{1}\left\{f_{n_1} > M\right\} f_{n_1} d\mu = 0.$$

We will show the stronger result that, for all $M > 0$,

$$\limsup_{N\to\infty} \int \mathbb{1}\left\{f_{n_1} > M\right\} f_{n_1} d\mu = 0. \tag{S.1.5}$$

Let $M > 0$ and $\nu \in (1, \infty)$ such that

$$\nu < \frac{b_1 + \delta - \delta_1}{b_1} \wedge \frac{b_2 + \delta - \delta_2}{b_2}. \tag{S.1.6}$$

By Hölder's inequality,

$$\int \mathbb{1}\left\{f_{n_1} > M\right\} f_{n_1} d\mu \le \left(\int f_{n_1}^{\nu} d\mu\right)^{1/\nu} \left(\int \mathbb{1}\left\{f_{n_1} > M\right\} d\mu\right)^{(\nu-1)/\nu}. \tag{S.1.7}$$

First, we show that the first term in the right-hand side of (S.1.7) is $O(1)$.

$$\int f_{n_1}^{\nu} d\mu = \int_0^1 E\left[\mathbb{1}\left\{\xi_{(1)} < t < \xi_{(n_1)}\right\} |\nu_{n_1}(t)|^{\nu}\right] q(t) dF_Y^{-1}(t).$$

Let $B(t) := C_U t^{-b_1}(1-t)^{-b_2}$. By the triangle inequality, Assumption 2, and since B^{ν} is convex and $u_{n_1}(t)$ lies between $\mathbb{G}_{n_1}(t)$ and t, for all $t \in (\xi_{(1)}, \xi_{(n_1)})$, almost surely,

$$\begin{aligned} |\nu_{n_1}(t)|^{\nu} &\le B(t)^{\nu} + B(u_{n_1}(t))^{\nu} \\ &\le 2B(t)^{\nu} + B(\mathbb{G}_{n_1}(t))^{\nu}. \end{aligned}$$

Hence,

$$\int f_{n_1}^{\nu} d\mu \lesssim \int_0^1 B(t)^{\nu} q(t) dF_Y^{-1}(t) + \int_0^1 E\left[\mathbb{1}\left\{\xi_{(1)} < t < \xi_{(n_1)}\right\} B(\mathbb{G}_{n_1}(t))^{\nu}\right] q(t) dF_Y^{-1}(t).$$

By Lemma 6, for ν sufficiently small, the first integral is finite and does not depend on n_1. Consider the second integral. By Lemma 6 again, to show that this integral is $O(1)$ it suffices to show that

$$v_{n_1}(t) := E\left[\mathbb{1}\left\{\xi_{(1)} < t < \xi_{(n_1)}\right\} B(\mathbb{G}_{n_1}(t))^{\nu}\right] q(t) \lesssim t^{1/2-b_1-\delta}(1-t)^{1/2-b_2-\delta}, \tag{S.1.8}$$

which we prove below. We have

$$\begin{aligned} v_{n_1}(t) &= E\left[\mathbb{1}\left\{1 < n_1\mathbb{G}_{n_1}(t) < n_1\right\} B\left(\left(n_1\mathbb{G}_{n_1}(t)\right)/n_1\right)^{\nu}\right] q(t) \\ &= C_U t^{1/2-\delta_1}(1-t)^{1/2-\delta_2} E\left[\mathbb{1}\left\{1 < W < n_1\right\} \left(\frac{W}{n_1}\right)^{-b_1\nu} \left(1 - \frac{W}{n_1}\right)^{-b_2\nu}\right] \\ &\lesssim t^{1/2-b_1-\delta}(1-t)^{1/2-b_2-\delta} w_{n_1}(t), \end{aligned}$$

where $W\sim$Binomial(n_1, t) and

$$w_{n_1}(t) = t^{b_1+\delta-\delta_1}(1-t)^{b_2+\delta-\delta_2} E\left[\mathbb{1}\left\{1 < W < n_1\right\}\left(\frac{W}{n_1}\right)^{-b_1\nu}\left(1-\frac{W}{n_1}\right)^{-b_2\nu}\right].$$

Hence, it suffices to show that $w_{n_1}(t) = O(1)$ uniformly in t. Since

$$\begin{aligned} &E\left[\mathbb{1}\left\{1 < W < n_1\right\}\left(\frac{W}{n_1}\right)^{-b_1\nu}\left(1-\frac{W}{n_1}\right)^{-b_2\nu}\right] \\ &\leq E^{1/2}\left[\mathbb{1}\left\{1 < W < n_1\right\}\left(\frac{W}{n_1}\right)^{-2b_1\nu}\right] E^{1/2}\left[\mathbb{1}\left\{1 < W < n_1\right\}\left(1-\frac{W}{n_1}\right)^{-2b_2\nu}\right], \end{aligned}$$

it suffices to show

$$\begin{aligned} E\left[\mathbb{1}\left\{1 < W < n_1\right\}\left(\frac{W}{n_1}\right)^{-2b_1\nu}\right] &\lesssim t^{-2b_1\nu}, \\ E\left[\mathbb{1}\left\{1 < W < n_1\right\}\left(1-\frac{W}{n_1}\right)^{-2b_2\nu}\right] &\lesssim (1-t)^{-2b_2\nu}. \end{aligned} \tag{S.1.9}$$

We show the first result only (the second can be obtained analogously). We have

$$\begin{aligned} &E\left[\mathbb{1}\left\{1 < W < n_1\right\}\left(\frac{W}{n_1}\right)^{-2b_1\nu}\right] \\ &= E\left[\mathbb{1}\left\{1 < W < n_1\right\}\mathbb{1}\left\{W \geq \frac{n_1 t}{2}\right\}\left(\frac{W}{n_1}\right)^{-2b_1\nu}\right] \\ &\quad + E\left[\mathbb{1}\left\{1 < W < n_1\right\}\mathbb{1}\left\{W < \frac{n_1 t}{2}\right\}\left(\frac{W}{n_1}\right)^{-2b_1\nu}\right] \\ &\lesssim t^{-2b_1\nu} + E\left[\mathbb{1}\left\{W < \frac{n_1 t}{2}\right\}\left(\frac{1}{n_1}\right)^{-2b_1\nu}\right] \\ &= t^{-2b_1\nu} + n_1^{2b_1\nu}\Pr\left(W < \frac{n_1 t}{2}\right). \end{aligned}$$

By Hoeffding's inequality, we have

$$\Pr\left(W < \frac{n_1 t}{2}\right) \leq \exp\left(-\frac{1}{8}n_1 t\right).$$

Since $x^\delta \exp(-x/8)$ is bounded on $[0, \infty)$ for any $\delta > 0$, we have

$$n_1^{2b_1\nu}\Pr\left(W < \frac{n_1 t}{2}\right) \lesssim t^{-2b_1\nu}$$

and thus

$$E\left[\mathbb{1}\left\{1 < W < n_1\right\}\left(\frac{W}{n_1}\right)^{-2b_1\nu}\right] \lesssim t^{-2b_1\nu},$$

which proves (S.1.9), $w_{n_1}(t) = O(1)$, (S.1.8), and in turn that the first term in (S.1.7) is $O(1)$.

Next, we show that the second term in (S.1.7) converges to zero. We have

$$\begin{aligned}\int \mathbb{1}\left\{f_{n_1} > M\right\} d\mu &= \int_0^1 \Pr\left(\mathbb{1}\left\{\xi_{(1)} < t < \xi_{(n_1)}\right\} |\nu_{n_1}(t)| > M\right) q(t) dF_Y^{-1}(t) \\ &\leq \int_0^1 \Pr\left(|\nu_{n_1}(t)| > M\right) q(t) dF_Y^{-1}(t). \end{aligned} \tag{S.1.10}$$

We have already shown that, for all $t \in (0,1)$, $|\nu_{n_1}(t)| \xrightarrow{\text{a.s.}} 0$. It follows that, for all $t \in (0,1)$, $|\nu_{n_1}|(t) = o_P(1)$. This implies that

$$\lim_{N\to\infty} \Pr\left(|\nu_{n_1}(t)| > M\right) q(t) = 0, \quad \forall t \in (0,1).$$

Moreover, for all $t \in (0,1)$, $\Pr\left(|\nu_{n_1}(t)| > M\right) q(t) \leq q(t)$ with $\int_0^1 q(t) dF_Y^{-1}(t) < \infty$ by Lemma 6. By the dominated convergence theorem,

$$\lim_{N\to\infty} \int_0^1 \Pr\left(|\nu_{n_1}(t)| > M\right) q(t) dF_Y^{-1}(t) = 0,$$

and (S.1.10) implies that (S.1.5) holds. This completes the proof that

$$E\left[\int_{\xi_{(1)}}^{\xi_{(n_1)}} |\nu_{n_1}(t)| q(t) dF_Y^{-1}(t)\right] = o(1),$$

and thus

$$\int_{\xi_{(1)}}^{\xi_{(n_1)}} |\nu_{n_1}(t)| q(t) dF_Y^{-1}(t) = o_P(1). \tag{S.1.11}$$

To prove $R_3 = o_P(1)$, it is thus sufficient to show that the first term in (S.1.4) is bounded in probability. By Equation (2) in Chapter 2, Section 7 (p. 141) in Shorack and Wellner (1986), there exists a Brownian bridge $\mathbb{U}$ such that

$$\sup_{t\in(0,1)} \left|\frac{\sqrt{n_1}(\mathbb{G}_{n_1}(t) - t) - \mathbb{U}(t)}{q(t)}\right| = o_P(1).$$

Since $||./q||$ is a norm, the triangular inequality yields

$$\sup_{t\in(0,1)} \left|\frac{\sqrt{n_1}(\mathbb{G}_{n_1}(t) - t)}{q(t)}\right| \leq \sup_{t\in(0,1)} |\mathbb{U}(t)/q(t)| + o_P(1) = O_P(1) + o_P(1) = O_P(1),$$

by inequality (17) p. 451 in Shorack and Wellner (1986) with $a = 0, b = 1/2$, and by noticing that $\mathbb{U}/q$ has the same distribution on $[0, 1/2]$ and $[1/2, 1]$, and that the integral on the right-hand side of the inequality is finite for $q(t) = [t(1-t)]^a$ with $a < 1/2$. This result, together with (S.1.11) and (S.1.4), implies that $R_3 = o_P(1)$.

Fourth, we show that $R_4 = o_P(1)$. We have

$$|R_4| \leq \sqrt{N} \int_0^{\xi_{(1)}} x d\Lambda(x) + \sqrt{N} \int_{\xi_{(n_1)}}^1 (1-x) d\Lambda(x).$$

Consider the first term (the second term can be controlled analogously). Let $\delta \in (b_1 + d_1, 1/2)$. With probability tending to 1, $\xi_{(1)} \leq 1/2$. Moreover, on this event, we have, by Assumption 2,

$$\sqrt{N} \int_0^{\xi_{(1)}} x d\Lambda(x) \lesssim \sqrt{N} \xi_{(1)}^{1-\delta} \int_0^{1/2} x^\delta d\Lambda(x)$$

$$\lesssim \sqrt{\frac{N}{n_1}}\, n_1^{\delta-1/2}\left[n_1\xi_{(1)}\right]^{1-\delta}\int_0^{1/2} x^{\delta-b_1}\, dF_Y^{-1}(x).$$

By Lemma 6, $\int_0^{1/2} x^{\delta-b_1} dF_Y^{-1}(x) < \infty$. The result follows from $[n_1\xi_{(1)}]^{1-\delta} = O_P(1)$, $\sqrt{N/n_1} \leq 1$, and $\delta < 1/2$.

□

Lemma S.1.2 *$E[|R_5|] \to 0$ as $N \to \infty$.*

Proof: For that purpose, let $I_{n_1}(x) = (x, \mathbb{G}_{n_1}(x)]$ if $\mathbb{G}_{n_1}(x) > x$, $I_{n_1}(x) = [\mathbb{G}_{n_1}(x), x)$ if $\mathbb{G}_{n_1}(x) < x$, and $\emptyset$ otherwise. Finally, let $S_{n_1}(x) = \text{sgn}(\mathbb{G}_{n_1}(x) - x)$. Observe first that

$$\mathbb{V}_{n_2} \circ F_U \circ \mathbb{G}_{n_1}(x) - \mathbb{V}_{n_2} \circ F_U(x) = S_{n_1}(x) Z_N(x), \tag{S.1.12}$$

with

$$Z_N(x) = \frac{1}{\sqrt{n_2}} \sum_{i=1}^{n_2} \left[\mathbb{1}\left\{U_i \in I_{n_1}(x)\right\} - P_U(I_{n_1}(x))\right],$$

where $P_U([a,b]) = P_U((a,b]) = P_U([a,b)) = P_U((a,b)) = F_U(a) - F_U(b)$ for all $(a,b) \in [0,1], a \leq b$. Then,

$$\begin{aligned}
E\left[|R_5| \,|(\xi_i)_i\right] &\leq E\left[\int_0^1 \left|\mathbb{V}_{n_2} \circ F_U \circ \mathbb{G}_{n_1} - \mathbb{V}_{n_2} \circ F_U\right| dF_Y^{-1} \middle| (\xi_i)_i\right] \\
&= \int_0^1 E\left[\left|\mathbb{V}_{n_2} \circ F_U \circ \mathbb{G}_{n_1} - \mathbb{V}_{n_2} \circ F_U\right| \,|(\xi_i)_i\right] dF_Y^{-1} \\
&\leq \int_0^1 E\left[Z_N(x)^2 |(\xi_i)_i\right]^{1/2} dF_Y^{-1}(x) \\
&= \int_0^1 V\left[\mathbb{1}\left\{U_1 \in I_{n_1}(x)\right\} |(\xi_i)_i\right]^{1/2} dF_Y^{-1}(x) \\
&\leq \int_0^1 |P_U(I_{n_1}(x))|^{1/2} dF_Y^{-1}(x).
\end{aligned} \tag{S.1.13}$$

The first equality follows by Fubini–Tonelli's theorem, the second inequality uses (S.1.12) and the Cauchy–Schwarz inequality, and the second equality holds since conditional on the $(\xi_i)_i$, the variables $\mathbb{1}\left\{U_i \in I_{n_1}(x)\right\} - P_U(I_{n_1}(x))$ are i.i.d. with mean zero. As a result,

$$\begin{aligned}
E\left[|R_5|\right] &\leq \int_0^1 E\left[|P_U(I_{n_1}(x)|^{1/2}\right] dF_Y^{-1}(x) \\
&\leq \int_0^1 E\left[|P_U(I_{n_1}(x)|\right]^{1/2} dF_Y^{-1}(x),
\end{aligned} \tag{S.1.14}$$

where the first inequality follows by (S.1.13) and Fubini–Tonelli's theorem, whereas the second is due to Jensen's inequality. Now, by the law of large numbers and the continuous mapping theorem, $|F_U(\mathbb{G}_{n_1}(x)) - F_U(x)| \xrightarrow{P} 0$ for all $x \in [0,1]$. Moreover, $|F_U(\mathbb{G}_{n_1}(x)) - F_U(x)| \leq 1$. Hence, for all $x \in [0,1]$,

$$E\left[|F_U(x) - F_U(\mathbb{G}_{n_1}(x))|\right] \to 0.$$

We now apply the dominated convergence theorem to prove that $E[|R_5|] \to 0$. Because $x \mapsto E[|P_U(I_{n_1}(x))|]^{1/2}$ is bounded by 1 for all n_1, it is actually enough to bound this function for

x close to 0 and close to 1. Also, by symmetry, we can focus without loss of generality on the neighborhood of 0. We prove that

$$E\left[|P_U(I_{n_1}(x))|\right] \lesssim x^{1-b_1}. \tag{S.1.15}$$

Then the result follows by Lemma 6 combined with Assumption 2(iv). To prove (S.1.15), we apply Lemma 7 with $Q_n(x) := \mathbb{G}_{n_1}(x)$, $B_n(x) = 1$, and $\delta < \exp(-1)/2$. If $x \geq 1/n_1$, the Cauchy–Schwarz inequality yields

$$E\left[|\mathbb{G}_{n_1}(x) - x|\right] \leq \left[\frac{x(1-x)}{n_1}\right]^{1/2} \leq 2x, \tag{S.1.16}$$

since $n_1^{1/2} \geq x^{-1/2}$. If $x < 1/n_1$, (S.1.16) holds as well by Theorem 1 in Berend and Kontorovich (2013). Hence, (S.1.16) holds for all $x \in (0, \bar{\delta}/2)$. Next, let $n_0 \in \mathbb{N}$, $n_0 \geq 4/(1-\bar{\delta})^2$. By Kiefer's inequality (see, e.g. Van der Vaart and Wellner, 1996, Corollary A.6.3), we have, for all $x \in [0, \delta]$ and all $n_1 \geq n_0$,

$$\Pr(\mathbb{G}_{n_1}(x) > 1/2) \leq (ex)^{n_1(1-\bar{\delta})^2/4} \lesssim x.$$

Thus, we can apply Lemma 7, which yields (S.1.15).

□

Lemma S.1.3 *For all $j \in \{6,7,8,9\}$: $R_j = o_P(1)$ as $N \to \infty$.*

Proof: First, we show that $R_6 = o_p(1)$. We split R_6 into three integrals: $R_{6,1}$ is obtained by integrating along the segment $[\xi_{(1)}, \zeta_{(1)} \vee \xi_{(1)}]$, $R_{6,2}$ is obtained by integrating along the segment $(\zeta_{(1)} \vee \xi_{(1)}, \zeta_{(n_3)} \wedge \xi_{(n_1)}]$, and $R_{6,3}$ is obtained by integrating on the segment $(\zeta_{(n_3)} \wedge \xi_{(n_1)}, \xi_{(n_1)}]$. Let us first focus on $R_{6,2}$. By the mean value theorem, for all $t \in [\zeta_{(1)} \vee \xi_{(1)}, \zeta_{(n_3)} \wedge \xi_{(n_1)}]$, there exists $u_N(t) \in (\mathbb{G}_{n_1}(t), \mathbb{H}_{n_3}^{-1} \circ \mathbb{G}_{n_1}(t))$ such that

$$R_{6,2} = \sqrt{\frac{N}{n_3}} \sqrt{n_3} \int_{\zeta_{(1)} \vee \xi_{(1)}}^{\zeta_{(n_3)} \wedge \xi_{(n_1)}} \underbrace{\left[f_U - f_U \circ u_N\right]}_{=:\nu_N} \left[\mathbb{H}_{n_3}^{-1} \circ \mathbb{G}_{n_1} - \mathbb{G}_{n_1}\right] dF_Y^{-1}.$$

By Assumption 2(iv), there exists $\delta > 0$ such that $b_j + d_j < 1/2 - \delta$. Further, let $\delta_j > 0$ be such that $\delta > 2(\delta_1 \vee \delta_2)$ and

$$b_j + d_j < 1/2 - \delta - \delta_j. \tag{S.1.17}$$

Then let $q(t) = t^{1/2-\delta_1}(1-t)^{1/2-\delta_2}$. From what precedes, we have

$$|R_{6,2}| \leq \sqrt{\frac{N}{n_3}} \sup_{t \in (1/n_1, 1-1/n_1)} \left| \frac{\sqrt{n_3}(\mathbb{H}_{n_3}^{-1}(t) - t)}{q(t)} \right| \int_{\zeta_{(1)} \vee \xi_{(1)}}^{\zeta_{(n_3)} \wedge \xi_{(n_1)}} |\nu_N(t)| q(t) dF_Y^{-1}(t). \tag{S.1.18}$$

We now show that the last term in (S.1.18) tends to 0 in L_1. Let

$$f_N(t) := \mathbb{1}\left\{\zeta_{(1)} \vee \xi_{(1)} < t < \zeta_{(n_3)} \wedge \xi_{(n_1)}\right\} |\nu_N(t)|,$$

and let μ denote the measure defined as the product of the probability measure on Ω, the probability space on which all random variables are defined, with the measure ρ on $(0,1)$ such that $d\rho/dF_Y^{-1} = q$. By Lemma 6, $\mu(\Omega \times (0,1)) = \int_0^1 q(t) dF_Y^{-1}(t) < \infty$. Moreoever,

$$E\left[\int_{\zeta_{(1)} \vee \xi_{(1)}}^{\zeta_{(n_3)} \wedge \xi_{(n_1)}} |\nu_N(t)| q(t) dF_Y^{-1}(t)\right] = \int_{\Omega \times (0,1)} f_N d\mu.$$

First, we show that f_N tends to 0 μ-almost surely. By convergence of $\mathbb{G}_{n_1}(t)$ and $\mathbb{H}_{n_3}^{-1}(t)$ to t, we have, for all $t \in (0,1)$, $u_N(t) \xrightarrow{\text{a.s.}} t$. Then, by continuity of f_U, $\nu_N(t) \xrightarrow{\text{a.s.}} 0$ for all $t \in (0,1)$. This shows that $f_N \to 0$ μ-almost-surely. Second, by Corollary(i)–(ii) of Theorem 16.14 in (Billingsley, 1995, p. 218), it is sufficient to show that f_N is uniformly integrable, i.e., that

$$\lim_{M\to\infty} \limsup_{N\to\infty} \int \mathbb{1}\left\{f_N > M\right\} f_N d\mu = 0. \tag{S.1.19}$$

We will show the stronger result that, for all $M > 0$,

$$\limsup_{N\to\infty} \int \mathbb{1}\left\{f_N > M\right\} f_N d\mu = 0. \tag{S.1.20}$$

Let $M > 0$ and $\nu \in (1, \infty)$ such that (S.1.6) holds. By Hölder's inequality,

$$\int \mathbb{1}\left\{f_N > M\right\} f_N d\mu \leq \left(\int f_N^\nu d\mu\right)^{1/\nu} \left(\int \mathbb{1}\left\{f_N > M\right\} d\mu\right)^{(\nu-1)/\nu}. \tag{S.1.21}$$

Consider the first term.

$$\int f_N^\nu d\mu = \int_0^1 E\left[\int_{\zeta_{(1)}\vee\xi_{(1)}}^{\zeta_{(n_3)}\wedge\xi_{(n_1)}} |\nu_N(t)|^\nu q(t)\right] dF_Y^{-1}(t).$$

Let $B(t) := C_U t^{-b_1}(1-t)^{-b_2}$. By Assumption 2 and since B^ν is convex and $u_N(t)$ lies between $\mathbb{G}_{n_1}(t)$ and t, for all $t \in (\xi_{(1)}, \xi_{(n_1)})$,

$$\begin{aligned}
|\nu_N(t)|^\nu &\leq 2^{\nu-1}[B(t)^\nu + B(u_N(t))^\nu] \\
&\leq 2^{\nu-1}[2B(t)^\nu + B(\mathbb{G}_{n_1}(t))^\nu + B(\mathbb{H}_{n_3}^{-1}(\mathbb{G}_{n_1}(t)))^\nu], \quad \text{a.s..}
\end{aligned}$$

Hence,

$$\begin{aligned}
\int f_N^\nu d\mu \lesssim& \int_0^1 B(t)^\nu q(t) dF_Y^{-1}(t) + \int_0^1 E\left[\mathbb{1}\left\{\xi_{(1)} < t < \xi_{(n_1)}\right\} B(\mathbb{G}_{n_1}(t))^\nu q(t)\right] dF_Y^{-1}(t) \\
&+ \int_0^1 E\left[\mathbb{1}\left\{\zeta_{(1)} \vee \xi_{(1)} < t < \zeta_{(n_3)} \wedge \xi_{(n_1)}\right\} B(\mathbb{H}_{n_3}^{-1}(\mathbb{G}_{n_1}(t)))^\nu q(t)\right] dF_Y^{-1}(t).
\end{aligned}$$

The first two integrals are uniformly bounded as already shown in the proof of Lemma S.1.1. Let us focus on the third integral.

$$\begin{aligned}
&\int_0^1 E\left[\mathbb{1}\left\{\zeta_{(1)} \vee \xi_{(1)} < t < \zeta_{(n_3)} \wedge \xi_{(n_1)}\right\} B(\mathbb{H}_{n_3}^{-1}(\mathbb{G}_{n_1}(t)))^\nu q(t)\right] dF_Y^{-1}(t) \\
&\leq \int_0^1 E\left[\zeta_{(1)}^{\delta/2+(1-\nu)b_1-\delta_1}(1-\zeta_{(n_3)})^{\delta/2+(1-\nu)b_2-\delta_2}\right] t^{1/2-\delta-b_1}(1-t)^{1/2-\delta-b_2} dF_Y^{-1}(t) \\
&= n_3^{(b_1+b_2)(\nu-1)+\delta_1+\delta_2-\delta} E\left[\left(n_3\zeta_{(1)}\right)^{\delta/2+(1-\nu)b_1-\delta_1}\left(n_3(1-\zeta_{(n_3)})\right)^{\delta/2+(1-\nu)b_2-\delta_2}\right] \\
&\quad \times \int_0^1 t^{1/2-\delta-b_1}(1-t)^{1/2-\delta-b_2} dF_Y^{-1}(t).
\end{aligned}$$

For ν sufficiently small, $n_3^{(b_1+b_2)(\nu-1)+\delta_1+\delta_2-\delta} \leq 1$ and $x \mapsto x^{\delta/2+(1-\nu)b_j-\delta_j}$ is concave. By the Cauchy–Schwarz and Jensen inequalities,

$$E\left[\left(n_3\zeta_{(1)}\right)^{\delta/2+(1-\nu)b_1-\delta_1}\left(n_3(1-\zeta_{(n_3)})\right)^{\delta/2+(1-\nu)b_2-\delta_2}\right]$$

$$\begin{aligned}&\leq \left(E\left[\left(n_3\zeta_{(1)}\right)^{2(\delta/2+(1-\nu)b_1-\delta_1)}\right]\right)^{1/2}\left(E\left[\left(n_3(1-\zeta_{(n_3)})\right)^{2(\delta/2+(1-\nu)b_2-\delta_2)}\right]\right)^{1/2}\\&\leq E\left[n_3\zeta_{(1)}\right]^{\delta/2+(1-\nu)b_1-\delta_1}E\left[n_3(1-\zeta_{(n_3)})\right]^{\delta/2+(1-\nu)b_2-\delta_2}\\&=\left(\frac{n_3}{n_3+1}\right)^{\delta/2+(1-\nu)b_1-\delta_1}\left(\frac{n_3}{n_3+1}\right)^{\delta/2+(1-\nu)b_2-\delta_2}\\&\leq 1.\end{aligned}$$

Combining all pieces together yields

$$\begin{aligned}&\int_0^1 E\left[\mathbb{1}\left\{\zeta_{(1)}\vee\xi_{(1)}<t<\zeta_{(n_3)}\wedge\xi_{(n_1)}\right\}B(\mathbb{H}_{n_3}^{-1}(\mathbb{G}_{n_1}(t)))^{\nu}q(t)\right]dF_Y^{-1}(t)\\&\qquad\leq\int_0^1 t^{1/2-\delta-b_1}(1-t)^{1/2-\delta-b_2}dF_Y^{-1}(t),\end{aligned}$$

which does not depend on N and is finite by Lemma 6. Next, let us show that the second term in (S.1.21) converges to zero.

$$\begin{aligned}\int\mathbb{1}\left\{f_N>M\right\}d\mu&=\int_0^1\Pr\left(\mathbb{1}\left\{\zeta_{(1)}\vee\xi_{(1)}<t<\zeta_{(n_3)}\wedge\xi_{(n_1)}\right\}|\nu_N(t)|>M\right)q(t)dF_Y^{-1}(t)\\&\leq\int_0^1\Pr\left(|\nu_N(t)|>M\right)q(t)dF_Y^{-1}(t).\end{aligned}\qquad\text{(S.1.22)}$$

Since we have already shown that, for all $t\in(0,1)$, $|\nu_N(t)|\xrightarrow{\text{a.s.}}0$, it follows that, for all $t\in(0,1)$, $|\nu_N(t)|\xrightarrow{P}0$. This implies that

$$\lim_{N\to\infty}\Pr\left(|\nu_N(t)|>M\right)q(t)=0,\quad\forall t\in(0,1).$$

Moreover, for all $t\in(0,1)$, $\Pr\left(|\nu_N(t)|>M\right)q(t)\leq q(t)$ with $\int_0^1 q(t)dF_Y^{-1}(t)<\infty$ by Lemma 6. By the dominated convergence theorem,

$$\lim_{N\to\infty}\int_0^1\Pr\left(|\nu_N(t)|>M\right)q(t)dF_Y^{-1}(t)=0,$$

and (S.1.22) implies that (S.1.20) holds. This completes the proof that

$$E\left[\int_{\zeta_{(1)}\vee\xi_{(1)}}^{\zeta_{(n_3)}\wedge\xi_{(n_1)}}|\nu_N(t)|q(t)dF_Y^{-1}(t)\right]=o(1),$$

and thus

$$\int_{\zeta_{(1)}\vee\xi_{(1)}}^{\zeta_{(n_3)}\wedge\xi_{(n_1)}}|\nu_N(t)|q(t)dF_Y^{-1}(t)\xrightarrow{P}0.\qquad\text{(S.1.23)}$$

Next, by Corollary 4.3.1 and Theorem 3.4 in Csörgő et al. (1986),

$$\sup_{t\in(1/n_3,1-1/n_3)}\left|\frac{\sqrt{n_3}(\mathbb{H}_{n_3}^{-1}(t)-t)}{q(t)}\right|=O_P(1).$$

Let $\varepsilon>0$. For N sufficiently large, $1/n_1>1/n_3-\varepsilon$. It follows that

$$\sup_{t\in(1/n_1,1-1/n_1)}\left|\frac{\sqrt{n_3}(\mathbb{H}_{n_3}^{-1}(t)-t)}{q(t)}\right|\leq\sup_{t\in(1/n_3-\varepsilon,1-1/n_3+\varepsilon)}\left|\frac{\sqrt{n_3}(\mathbb{H}_{n_3}^{-1}(t)-t)}{q(t)}\right|$$

$$
\begin{aligned}
&= \sup_{t\in(1/n_3,1-1/n_3)} \left| \frac{\sqrt{n_3}(\mathbb{H}_{n_3}^{-1}(t)-t)}{q(t)} \right| + O_P(1) \\
&= O_P(1).
\end{aligned}
$$

This, together with (S.1.18), $\sqrt{\frac{N}{n_3}} = O_P(1)$, and (S.1.23), implies that $R_{6,2} = o_P(1)$. Now, let us show that $R_{6,1} = o_P(1)$.

$$
\begin{aligned}
|R_{6,1}| &:= \left| \sqrt{\frac{N}{n_3}} \sqrt{n_3} \int_{\xi_{(1)}}^{\zeta_{(1)}\vee\xi_{(1)}} A_N(t) \left[\mathbb{H}_{n_3}^{-1} \circ \mathbb{G}_{n_1}(t) - \mathbb{G}_{n_1}(t) \right] dF_Y^{-1}(t) \right|, \\
&\leq \sqrt{\frac{N}{n_3}} \left| F_Y(\zeta_{(1)}) - F_Y(\xi_{(1)}) \right| \mathbb{1}\left\{ \xi_{(1)} < \zeta_{(1)} \right\} \\
&\quad \times \sqrt{n_3}(\zeta_{(j_N)} + n_1^{-1}) \sup_{\xi_{(1)}<t<\zeta_{(1)}} \left[2B(t) + B(\mathbb{G}_{n_1}(t)) + B(\mathbb{H}_{n_3}^{-1}(\mathbb{G}_{n_1}(t))) \right] \\
&\leq O_P(1) o_P(1) \times n_1^{b_1-1/2} \left((n_1/n_3)^{1/2} \left[n_3\zeta_{(j_N)} \right] + (n_1/n_3)^{-1/2} \right) \left(\left[n_1\xi_{(1)} \right]^{-b_1} + 1 + (n_1/n_3)^{-b_1} \left[n_3\zeta_{(1)} \right]^{-b_1} \right)
\end{aligned}
$$

where $j_N = \lceil n_1/n_3 \rceil$. By assumption, there exists $\varepsilon > 0$ such that, for N sufficiently large, $|\frac{n_1}{n_3} - \frac{\lambda_3}{\lambda_1}| < \varepsilon$. By choosing ε sufficiently small, this ensures that for N sufficiently large N, $j_N \leq \lceil \varepsilon + \lambda_3/\lambda_1 \rceil \leq \lceil \lambda_3/\lambda_1 \rceil + 1 =: \bar{j}$. In particular, $n_3\zeta_{(j_N)} \leq n_3\zeta_{(\bar{j})} = O_P(1)$. Since $[n_1\xi_{(1)}]^{-b_1} = O_P(1)$ and $[n_3\zeta_{(1)}]^{-b_1} = O_P(1)$, and $b_1 < 1/2$ by Assumption 2(iv),

$$
R_{6,1} \leq O_P(1) o_P(1) o_P(1) O_P(1) = o_P(1).
$$

An analogous reasoning yields $R_{6,3} = o_P(1)$.

Second, we show that R_7 is defined in (22). We actually prove the stronger result that R_7 converges to 0 in L^1. Let $\mathbb{W}_{n_3} = \sqrt{n_3}(\mathbb{H}_{n_3}^{-1} - \mathrm{I})$ and $B_{n_1} = \mathbb{1}\left\{ \xi_{(1)} \leq x < \xi_{(n_1)} \right\}$. We have, by Fubini–Tonelli's theorem,

$$
\begin{aligned}
E[|R_7|] &\leq \sqrt{\frac{N}{n_3}} \int_0^1 E[|\mathbb{W}_{n_3} \circ \mathbb{G}_{n_1}(x) - \mathbb{W}_{n_3}(x)| \times B_{n_1}] f_U(x) dF_Y^{-1}(x) \\
&\leq \sqrt{\frac{N}{n_3}} \int_0^1 E[(\mathbb{W}_{n_3} \circ \mathbb{G}_{n_1}(x) - \mathbb{W}_{n_3}(x))^2 \times B_{n_1}]^{1/2} f_U(x) dF_Y^{-1}(x).
\end{aligned}
$$

We apply the dominated convergence theorem to prove the result. First, note that for all $(x, y) \in (0, 1]^2$,

$$
|\mathbb{H}_{n_3}^{-1}(x) - \mathbb{H}_{n_3}^{-1}(y)| \sim \mathrm{Beta}(|\lceil n_3x \rceil - \lceil n_3y \rceil|, n_3 - |\lceil n_3x \rceil - \lceil n_3y \rceil| + 1),
$$

with the convention that the $\mathrm{Beta}(0, n_3 + 1)$ is the Dirac distribution at 0. Hence, for any $k \in \{1, \ldots, n_1 - 1\}$,

$$
\begin{aligned}
&E\left[(\mathbb{W}_{n_3} \circ \mathbb{G}_{n_1}(x) - \mathbb{W}_{n_3}(x))^2 \, | \mathbb{G}_{n_1}(x) = k/n_1 \right] \\
&= n_3 \left\{ E\left[\left(\mathbb{H}_{n_3}^{-1}(k/n_1) - \mathbb{H}_{n_3}^{-1}(x) - (k/n_1 - x) \right)^2 \right] \right\} \\
&= n_3 \left\{ E\left[\left(\mathbb{H}_{n_3}^{-1}(k/n_1) - \mathbb{H}_{n_3}^{-1}(x) - \frac{1}{n_3 + 1} \left(\lceil (n_3k)/n_1 \rceil - \lceil n_3x \rceil \right) \right. \right. \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{n_3+1}\left(\lceil (n_3k)/n_1\rceil - \lceil n_3x\rceil\right) - (k/n_1 - x)\Big)^2\Big]\Big]\Big\}\\
&= n_3\left\{V\left[\mathbb{H}_{n_3}^{-1}(k/n_1) - \mathbb{H}_{n_3}^{-1}(x)\right] + \frac{1}{(n_3+1)^2}(\lceil (n_3k)/n_1\rceil - (n_3k)/n_1 + n_3x - \lceil n_3x\rceil + x - k/n_1)^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{n_3}{(n_3+1)^2(n_3+2)}\left|\lceil (n_3k)/n_1\rceil - \lceil n_3x\rceil\right|\left(n_3+1-\left|\lceil (n_3k)/n_1\rceil - \lceil n_3x\rceil\right|\right)\\
&\quad + \frac{n_3}{(n_3+1)^2}(\lceil (n_3k)/n_1\rceil - (n_3k)/n_1 + n_3x - \lceil n_3x\rceil + x - k/n_1)^2
\end{aligned}
\tag{S.1.24}
$$

$$
\begin{aligned}
&\leq \frac{1}{n_3}\left|\lceil (n_3k)/n_1\rceil - \lceil n_3x\rceil\right| + \frac{2}{n_3}\left[2\left(\lceil (n_3k)/n_1\rceil - (n_3k)/n_1\right)^2 + 2\left(\lceil n_3x\rceil - n_3x\right)^2 + \left(\frac{k}{n_1}-x\right)^2\right]\\
&\leq \left|\frac{k}{n_1}-x\right| + \frac{1}{n_3}\left[10 + 2\left(\frac{k}{n_1}-x\right)^2\right],
\end{aligned}
\tag{S.1.25}
$$

where the first inequality follows by convexity and the last by the triangle inequality and because by definition, $|a - \lceil a\rceil| \leq 1$ for all $a \in \mathbb{R}_+$. Now, remark that $B_{n_1} = 1$ iff $n_1\mathbb{G}_{n_1}(x) \in \{1, \dots, n_1 - 1\}$. Then, by what precedes,

$$
\begin{aligned}
E\left[(\mathbb{W}_{n_3}\circ\mathbb{G}_{n_1}(x) - \mathbb{W}_{n_3}(x))^2 \times B_{n_1}\right] &\leq E\left[|\mathbb{G}_{n_1}(x) - x|\right] + \frac{1}{n_3}\left[10 + 2V(\mathbb{G}_{n_1}(x))\right]\\
&\to 0.
\end{aligned}
\tag{S.1.26}
$$

To apply the dominated convergence theorem, we show that there exists $q(.)$ such that for all $n_1 \geq n_0$ and all $x \in [0,1]$,

$$
E[(\mathbb{W}_{n_3}\circ\mathbb{G}_{n_1}(x) - \mathbb{W}_{n_3}(x))^2 \times B_{n_1}]^{1/2} \leq q(x),
\tag{S.1.27}
$$

with $\int_0^1 q(x)f_U(x)dF_Y^{-1}(x) < \infty$. As above, we focus on a neighborhood of 0. If $x > 1/n_3$, we have, by (S.1.26) and (S.1.16),

$$
E\left[(\mathbb{W}_{n_3}\circ\mathbb{G}_{n_1}(x) - \mathbb{W}_{n_3}(x))^2 \times B_{n_1}\right] \leq 14x.
$$

Now suppose that $x < 1/n_3$. Remark that $E(B_{n_1}) \leq 1 - (1-x)^{n_1} \leq n_1x$. Then, integrating (S.1.25), we obtain

$$
\begin{aligned}
E\left[(\mathbb{W}_{n_3}\circ\mathbb{G}_{n_1}(x) - \mathbb{W}_{n_3}(x))^2 \times B_{n_1}\right] &\leq E\left[|\mathbb{G}_{n_1}(x) - x|\right] + \frac{1}{n_3}\left[10n_1x + 2V(\mathbb{G}_{n_1}(x))\right]\\
&\leq (\lceil \lambda_3/\lambda_1\rceil + 1)14x.
\end{aligned}
$$

Then we can choose $q(x) = ((\lceil \lambda_3/\lambda_1\rceil + 1)14x)^{1/2}$ in (S.1.27). By Assumption 2 and Lemma 6, we have $\int_0^{1/2} q(x)f_U(x)dF_Y^{-1}(x) < \infty$. The same reasoning applies to the interval $[1/2, 1]$. The result follows.

Third, we show that $R_8 = o_P(1)$. Let Λ denote the measure on $(0,1)$ such that $d\Lambda/dF_Y^{-1} = f_U$. Note that $\mathbb{H}_{n_3}^{-1}(x) \sim$Beta$(\lceil n_3x\rceil, n_3 + 1 - \lceil n_3x\rceil)$, thus $E[\mathbb{H}_{n_3}^{-1}(x)] = \lceil n_3x\rceil/(n_3+1)$. Then

$$
E[|R_8|] \leq \sqrt{\frac{N}{n_3}}\int_0^1 [1 - x^{n_1} - (1-x)^{n_1}]\left|\frac{\lceil n_3x\rceil - (n_3+1)x}{(n_3+1)n_3^{-1/2}}\right| d\Lambda(x).
$$

Let $f_N(x)$ denote the integrand. We have $\lim_{N\to\infty} f_N(x) = 0$. Moreover, using $1-x^{n_1}-(1-x)^{n_1} \leq n_1 x$ and since $n_1 \lesssim n_3$, we obtain, when $x < 1/n_3$,

$$f_N(x) \leq 2n_3^{1/2} x \leq x^{1/2} \lesssim [x(1-x)]^{1/2}.$$

When $x \in [1/n_3, 1-1/n_3]$,

$$f_N(x) \leq \frac{2}{n_3^{1/2}} \lesssim [x(1-x)]^{1/2}.$$

Finally, when $x > 1-1/n_3$, then $x > 1-1/n_1$, and thus using $1-x^{n_1} \leq n_1(1-x)$,

$$f_N(x) \leq n_1(1-x)\frac{2}{(n_3+1)n_3^{-1/2}} \leq 2(1-x)^{1/2} \lesssim [x(1-x)]^{1/2}.$$

Moreover, $\int_0^1 [x(1-x)]^{1/2} d\Lambda < \infty$ by Lemma 6. Thus, by the dominated convergence theorem, $R_8 = o_P(1)$.

Fourth, we show that $R_9 = o_P(1)$. We prove the stronger result that R_9 converges to 0 in L^1. By Fubini–Tonelli's theorem combined with Assumption 1(ii) and Jensen's inequality, we have

$$\begin{aligned} E[|R_9|] \leq \sqrt{\frac{N}{n_3}} \int_0^1 & \sqrt{n_3} E\left[\left|\mathbb{1}\left\{x \in [\xi_{(1)}, \xi_{(n_1)}]\right\} - \mathbb{1}\left\{x \in [1/n_3, (n_3-1)/n_3]\right\}\right|\right] \\ & \times E\left[\left(\mathbb{H}_{n_3}^{-1}(x) - \frac{\lceil n_3 x\rceil}{(n_3+1)}\right)^2\right]^{1/2} d\Lambda(x). \end{aligned} \tag{S.1.28}$$

Since $\mathbb{H}_{n_3}^{-1}(x) \sim$Beta$(\lceil n_3 x\rceil, n_3+1-\lceil n_3 x\rceil)$, we have

$$E\left[\left(\mathbb{H}_{n_3}^{-1}(x) - \frac{\lceil n_3 x\rceil}{(n_3+1)}\right)^2\right]^{1/2} = \sqrt{\frac{\lceil n_3 x\rceil(n_3+1-\lceil n_3 x\rceil)}{(n_3+1)^2(n_3+2)}} \lesssim \sqrt{\frac{x(1-x)}{n_3}}.$$

Let $q_N(x)$ denote the first expectation in the integrand in (S.1.28). By letting $p_{n_1}(x) := 1-x^{n_1} - (1-x)^{n_1}$, we have

$$\begin{aligned} q_N(x) &= \Pr(\xi_{(1)} \leq x \leq \xi_{(n_1)}, x < 1/n_3) + \Pr(\xi_{(1)} \leq x \leq \xi_{(n_1)}, x > (n_3-1)/n_3) \\ &\quad + \Pr(\xi_{(1)} > x \cup x > \xi_{(n_1)}, 1/n_3 \leq x \leq (n_3-1)/n_3) \\ &= p_{n_1}(x)\left[\mathbb{1}\{x < 1/n_3\} + \mathbb{1}\{1-x < 1/n_3\}\right] + (1-p_{n_1}(x))\mathbb{1}\{1/n_3 \leq x \leq (n_3-1)/n_3\}. \end{aligned}$$

Let $f_N(x)$ denote the integrand in the right-hand side of (S.1.28). For all $x \in (0,1)$, $\lim_{N\to\infty} p_{n_1}(x) = 1$ so from what precedes, $\lim_{N\to\infty} f_N(x) = 0$ for all $x \in (0,1)$. Moreover, using $q_N(x) \leq 1$, we get

$$f_N(x) \lesssim [x(1-x)]^{1/2},$$

with $\int_0^1 [x(1-x)]^{1/2} d\Lambda < \infty$ by Lemma 6. The result follows by the dominated convergence theorem. □

Lemma S.1.4 *As* $N \to \infty$*:*

$$\sqrt{n_3} J_2 \xrightarrow{d} \mathcal{N}(0, \varsigma^2),$$

where

$$\varsigma^2 = \int_0^1 \int_0^1 [s \wedge t - st]\, f_U(s) f_U(t)\, dF_Y^{-1}(s)\, dF_Y^{-1}(t).$$

Proof: Let $I_{in_3} = [(i-1)/n_3, i/n_3)$. First, note that

$$-\sqrt{n_3} J_2 = \sum_{i=1}^{n_3} a_{in_3} \left(\zeta_{(i)} - \frac{i}{(n_3+1)} \right), \tag{S.1.29}$$

where $a_{1n_3} = a_{n_3 n_3} = 0$, and, for all $i \in \{2, \ldots, n_3 - 1\}$, $a_{in_3} = \sqrt{n_3} \Lambda(I_{in_3})$. We now verify that the necessary and sufficient conditions given by Hecker (1976) for the asymptotic normality of the L-statistic in (S.1.29) hold in our case. Let us define

$$\sigma_{n_3}^2 = \frac{1}{n_3+2} \sum_{j=1}^{n_3} \sum_{k=1}^{n_3} a_{jn_3} a_{kn_3} \left[\left(\frac{j}{n_3+1} \wedge \frac{k}{n_3+1} \right) - \frac{jk}{(n_3+1)^2} \right].$$

We have to prove that

$$\lim_{n_3 \to \infty} \frac{\max_{1 \le i \le n_3} \left| \sum_{j=i}^{n_3} a_{jn_3} \right|}{n_3 \sigma_{n_3}} = 0. \tag{S.1.30}$$

First, by Assumption 2 and Lemma 6, there exists $\delta < 1/2$ such that

$$\int_0^1 t^{\delta - b_1} (1-t)^{\delta - b_2} dF_Y^{-1}(t) < \infty.$$

Now, because $a_{in_3} \ge 0$, we have, for all $n_3 \ge 2$,

$$\begin{aligned}
\max_{1 \le i \le n_3} \left| \sum_{j=i}^{n_3} a_{jn_3} \right| &= \sqrt{n_3} \sum_{j=2}^{n_3-1} \Lambda(I_{jn_3}) \\
&= \sqrt{n_3} \int_{1/n_3}^{(n_3-1)/n_3} f_U(t) dF_Y^{-1}(t) \\
&\le C_U \sqrt{n_3} \int_{1/n_3}^{(n_3-1)/n_3} t^{-b_1} (1-t)^{-b_2} dF_Y^{-1}(t) \\
&\le C_U 2^\delta n_3^{1/2+\delta} \int_{1/n_3}^{(n_3-1)/n_3} t^{\delta - b_1} (1-t)^{\delta - b_2} dF_Y^{-1}(t) \\
&\le C_U 2^\delta n_3^{1/2+\delta} \int_0^1 t^{\delta - b_1} (1-t)^{\delta - b_2} dF_Y^{-1}(t),
\end{aligned}$$

where the first inequality follows by Assumption 2 and the second uses the fact that $[t(1-t)]^\delta \ge 1/(2n_3)^\delta$ for all $t \in [1/n_3, 1 - 1/n_3]$. Therefore,

$$\max_{1 \le i \le n} \left| \sum_{j=i}^{n_3} a_{jn_3} \right| = O(n_3^{1/2+\delta}). \tag{S.1.31}$$

Next, we have

$$\begin{aligned}
\sigma_{n_3}^2 &= \frac{n_3}{n_3+2} \sum_{j=2}^{n_3-1} \sum_{k=2}^{n_3-1} \Lambda(I_{jn_3}) \Lambda(I_{kn_3}) \left(\frac{j}{n_3+1} \wedge \frac{k}{n_3+1} - \frac{jk}{(n_3+1)^2} \right) \\
&= \frac{n}{n_3+2} \int_0^1 \int_0^1 f_{n_3}(x,y) d\Lambda(x) d\Lambda(y),
\end{aligned}$$

where $f_{n_3}(x,y) = \frac{j}{n_3+1} \wedge \frac{k}{n_3+1} - \frac{jk}{(n_3+1)^2}$ when $(x,y) \in I_{jn_3} \times I_{kn_3}$, $1 < j \wedge k \leq j \vee k < n_3$, $f_{n_3}(x,y) = 0$ otherwise. For any $(x,y) \in (0,1)^2$, $f_{n_3}(x,y) \to f(x,y) := x \wedge y - xy$. Moreover, for any $(x,y) \in I_{jn_3} \times I_{kn_3}$, $1 < j \wedge k \leq j \vee k < n_3$,

$$\begin{aligned} \frac{j}{n_3+1} \wedge \frac{k}{n_3+1} &\leq 2(x \wedge y), \\ 1 - \frac{j}{n_3+1} \vee \frac{k}{n_3+1} &\leq 2\left(1 - x \vee y\right). \end{aligned}$$

Thus, $f_{n_3}(x,y) \leq 4f(x,y)$ for all $(x,y) \in [1/n_3, 1-1/n_3]^2$. This inequality also holds for $(x,y) \in [0,1]^2 \backslash [1/n_3, 1-1/n_3]^2$ since $f_{n_3}(x,y) = 0$ for such (x,y). Because $x \wedge y \leq (xy)^{1/2}$ and $1 - x \vee y \leq [(1-x)(1-y)]^{1/2}$, we have $f(x,y) \leq [x(1-x)y(1-y)]^{1/2}$. Moreover, by Lemma 6, $\int_0^1 [\mathrm{I}(1-\mathrm{I})]^{1/2} d\Lambda < \infty$. Thus, by the dominated convergence theorem,

$$\lim_{n_3 \to \infty} \sigma_{n_3}^2 = \sigma^2 := \int_0^1 \int_0^1 (x \wedge y - xy) d\Lambda(x) d\Lambda(y) > 0. \tag{S.1.32}$$

Combined with (S.1.31), this implies (S.1.30). Thus, by Theorem 1 of Hecker (1976) and (S.1.32) again,

$$-\sqrt{n_3} J_2 \xrightarrow{d} \mathcal{N}(0, \varsigma^2).$$

□

S.2 Lemmas for Theorem 2

Lemma S.2.1 *For all $k \in \{1,2,3,4,5\}$: $J_k = o(1)$ as $N \to \infty$.*

Proof: First, we show that $J_1 = o(1)$. By sub-additivity of $u \mapsto |u|^\beta$ on $\mathbb{R}$ (since $\beta \in (0,1]$) and using that $|\widetilde{F}_Y^{(1)}(y) - \widehat{F}_Y^{(1)}(y)| \leq h_{n_2, \widehat{F}_Y^{(1)}(y)}$, we have

$$\begin{aligned} J_1 &\leq 2c_U E\left[\int_{\mathbb{R}^2} \left(\left|\widehat{F}_Y^{(1)}(y) - F_Y(y)\right|^\beta\right) \left(\Omega \circ \widehat{F}_Y^{(1)} \otimes \widehat{F}_Y^{(2)}\right)(y,y')\, dydy'\right] \\ &\quad + 2c_U E\left[\int_{\mathbb{R}^2} \left(\left|h_{n_2, \widehat{F}_Y^{(1)}(y)}\right|^\beta\right) \left(\Omega \circ \widehat{F}_Y^{(1)} \otimes \widehat{F}_Y^{(2)}\right)(y,y')\, dydy'\right] \\ &= 2c_U E\left[\int_{\mathbb{R}^2} \left(\left|\widehat{F}_Y^{(1)}(y) - F_Y(y)\right|^\beta\right) \left(\Omega \circ \widehat{F}_Y^{(1)} \otimes \widehat{F}_Y^{(2)}\right)(y,y')\, dydy'\right] \\ &\quad + O(\varepsilon_{n_2}) E\left[\int_{\mathbb{R}^2} \left(\Omega \circ \widehat{F}_Y^{(1)} \otimes \widehat{F}_Y^{(2)}\right)\right]. \end{aligned} \tag{S.2.1}$$

The second term tends to zero since $\varepsilon_{n_2} \to 0$ and $E\left[\int_{\mathbb{R}^2} \left(\Omega \circ \widehat{F}_Y^{(1)} \otimes \widehat{F}_Y^{(2)}\right)\right] < \infty$ by Lemma 9. We now focus on the first term in (S.2.1). Letting $\delta \in (0,1)$ be such that $\delta \leq 1/2 - (b_1 + d_1) \vee (b_2 + d_2)$ (using Assumption 2(iv)), we have

$$\begin{aligned} &E\int_{\mathbb{R}^2} \left(\left|\widehat{F}_Y^{(1)}(y) - F_Y(y)\right|^\beta\right) \left(\Omega \circ \widehat{F}_Y^{(1)} \otimes \widehat{F}_Y^{(2)}\right)(y,y')\, dydy' \\ &= E\int_{\mathbb{R}^2} \left(\left|\widehat{F}_Y^{(1)}(y) - F_Y(y)\right|^\beta\right) \left(\widehat{F}_Y^{(1)}(y) \wedge \widehat{F}_Y^{(2)}(y')\right)^{1-2b_1} \left(\bar{\widehat{F}}_Y^{(1)}(y) \wedge \bar{\widehat{F}}_Y^{(2)}(y')\right)^{1-2b_2} dydy' \end{aligned}$$

$$
\begin{aligned}
&\leq E\left[\int_{\mathbb{R}} \left|\widehat{F}_Y^{(1)}(y) - F_Y(y)\right|^{\beta} \left(\widehat{F}_Y^{(1)}(y)\right)^{d_1} \left(\bar{\widehat{F}}_Y^{(1)}(y)\right)^{d_2} \mathbb{1}_{\left\{\widehat{F}_Y^{(1)}(y)\bar{\widehat{F}}_Y^{(1)}(y)\neq 0\right\}} dy\right] \\
&\quad \times E\left[\int_{\mathbb{R}} \left(\widehat{F}_Y^{(2)}(y)\right)^{d_1+\delta} \left(\bar{\widehat{F}}_Y^{(2)}(y)\right)^{d_2+\delta} dy\right] \\
&\leq \int_{\mathbb{R}} E^{1/2}\left[\left|\widehat{F}_Y^{(1)}(y) - F_Y(y)\right|^{2\beta} \mathbb{1}_{\left\{\widehat{F}_Y^{(1)}(y)\bar{\widehat{F}}_Y^{(1)}(y)\neq 0\right\}}\right] E^{1/2}\left[\left(\widehat{F}_Y^{(1)}(y)\right)^{2d_1} \left(\bar{\widehat{F}}_Y^{(1)}(y)\right)^{2d_2}\right] dy \times C \quad \text{by Lemma 9} \\
&\leq 2^{\beta/2} C \int_{\mathbb{R}} \left(\frac{F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{\beta/2} E^{1/2}\left[\left(\widehat{F}_Y^{(1)}(y)\right)^{2d_1} \left(\bar{\widehat{F}}_Y^{(1)}(y)\right)^{2d_2}\right] dy \quad \text{by Lemma S.3.2} \\
&\leq 2^{\beta/2} C \int_{\mathbb{R}} \left(\frac{F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{\beta/2} \left(F_Y(y)\right)^{d_1} \left(\bar{F}_Y(y)\right)^{d_2} dy \quad \text{by concavity of } x \mapsto x^{2d_1}(1-x)^{2d_2} \\
&= o(1) \qquad \text{by Lemma 6.}
\end{aligned}
$$

Note that we used Lemma S.3.2 with $\tilde{b}_1 = (1-\beta)/2$. This yields $J_1 \to 0$.

Second, we show that $J_2 = o(1)$. In the analysis of this term, we will write

$$
\begin{aligned}
\alpha_1 &= 1 - 2b_1 \\
\alpha_2 &= 1 - 2b_2 \\
m(s,t) &= s \wedge t, \qquad \forall \alpha, s, t \in [0,1].
\end{aligned}
$$

It follows that $\Omega(s,t) = m(s,t)^{\alpha_1}\, m(\bar{s},\bar{t})^{\alpha_2}$. Then, by Assumption 2(iii), there exists a positive constant C_U such that $g \leq C_U^2$. We have

$$
\begin{aligned}
J_2 &= E\int_{\mathbb{R}^2} \left(g \circ F_Y^{\otimes 2}\right) \left|\left(\Omega \circ \widehat{F}_Y^{(1)} \otimes \widehat{F}_Y^{(2)}\right) - \left(\Omega \circ F_Y^{\otimes 2}\right)\right| \\
&\lesssim E\int_{\mathbb{R}^2} \left|\left(\Omega \circ \widehat{F}_Y^{(1)} \otimes \widehat{F}_Y^{(2)}\right) - \left(\Omega \circ F_Y^{\otimes 2}\right)\right| \\
&= E\int_{\mathbb{R}^2} \left|\left(m \circ \widehat{F}_Y^{(1)} \otimes \widehat{F}_Y^{(2)}\right)^{\alpha_1} \left(m \circ \bar{\widehat{F}}_Y^{(1)} \otimes \bar{\widehat{F}}_Y^{(2)}\right)^{\alpha_2} - \left(m \circ F_Y^{\otimes 2}\right)^{\alpha_1} \left(m \circ \bar{F}_Y^{\otimes 2}\right)^{\alpha_2}\right| \\
&\leq E\int_{\mathbb{R}^2} \left| \left(m \circ \widehat{F}_Y^{(1)} \otimes \widehat{F}_Y^{(2)}\right)^{\alpha_1} - \left(m \circ F_Y^{\otimes 2}\right)^{\alpha_1} \right| \left(m \circ \bar{\widehat{F}}_Y^{(1)} \otimes \bar{\widehat{F}}_Y^{(2)}\right)^{\alpha_2} \\
&\quad + E\int_{\mathbb{R}^2} \left(m \circ F_Y^{\otimes 2}\right)^{\alpha_1} \left|\left(m \circ \bar{\widehat{F}}_Y^{(1)} \otimes \bar{\widehat{F}}_Y^{(2)}\right)^{\alpha_2} - \left(m \circ \bar{F}_Y^{\otimes 2}\right)^{\alpha_2}\right| \\
&\leq E\int_{\mathbb{R}^2} \left| \left(m \circ \widehat{F}_Y^{(1)} \otimes \widehat{F}_Y^{(2)}\right)^{\alpha_1} - \left(m \circ F_Y^{\otimes 2}\right)^{\alpha_1} \right| \left(m \circ \bar{F}_Y^{\otimes 2}\right)^{\alpha_2} \\
&\quad + E\int_{\mathbb{R}^2} \left|\left(m \circ \bar{\widehat{F}}_Y^{(1)} \otimes \bar{\widehat{F}}_Y^{(2)}\right)^{\alpha_2} - \left(m \circ \bar{F}_Y^{\otimes 2}\right)^{\alpha_2}\right| \left(m \circ F_Y^{\otimes 2}\right)^{\alpha_1} \\
&\quad + E\int_{\mathbb{R}^2} \left| \left(m \circ \widehat{F}_Y^{(1)} \otimes \widehat{F}_Y^{(2)}\right)^{\alpha_1} - \left(m \circ F_Y^{\otimes 2}\right)^{\alpha_1} \right| \left| \left(m_{\alpha_2} \circ \bar{\widehat{F}}_Y^{(1)} \otimes \bar{\widehat{F}}_Y^{(2)}\right)^{\alpha_2} - \left(m \circ \bar{F}_Y^{\otimes 2}\right)^{\alpha_2} \right| \\
&=: A + A' + B.
\end{aligned}
$$

To show that $J_2 = o(1)$, it suffices to show that $A, A', B = o(1)$. Since the terms A and A' can be analyzed in a similar fashion, we only show that $A, B = o(1)$.

Term A: We have

$$A = \int_{\mathbb{R}^2} E\left(\left|(\widehat{F}_Y^{(1)}(y) \wedge \widehat{F}_Y^{(2)}(y'))^{\alpha_1} - (F_Y(y) \wedge F_Y(y'))^{\alpha_1}\right|\right)\left(\bar{F}_Y(y) \wedge \bar{F}_Y(y')\right)^{\alpha_2} dydy'.$$

By Lemma S.3.4, the term $E\left(\left|(\widehat{F}_Y^{(1)}(y) \wedge \widehat{F}_Y^{(2)}(y'))^{\alpha_1} - (F_Y(y) \wedge F_Y(y'))^{\alpha_1}\right|\right)$ goes to 0 as $N \to \infty$ for any fixed $y, y' \in \mathbb{R}$, and is dominated by the quantity $6\big(F_Y(y) \wedge F_Y(y')\big)^{\alpha_1}$ independently of N. Moreover,

$$\begin{aligned}
&\int_{\mathbb{R}^2} (F_Y(y) \wedge F_Y(y'))^{\alpha_1}\left(\bar{F}_Y(y) \wedge \bar{F}_Y(y')\right)^{\alpha_2} dydy' \\
&\leq \int_{\mathbb{R}^2} (F_Y(y)F_Y(y'))^{\alpha_1/2}(\bar{F}_Y(y)\bar{F}_Y(y'))^{\alpha_2/2} dydy' \\
&= \left(\int_{\mathbb{R}^2} F_Y(y)^{\alpha_1/2}\bar{F}_Y(y)^{\alpha_2/2}\, dy\right)^2.
\end{aligned}$$

The latter integral converges since $\alpha_1/2 > d_1$ and $\alpha_2/2 > d_2$ by Assumption 2(iv). By the dominated convergence theorem, we obtain that $A \to 0$ as $N \to \infty$.

Term B_N: By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
B = E&\int_{\mathbb{R}^2} \left|\left(\widehat{F}_Y^{(1)}(y) \wedge \widehat{F}_Y^{(2)}(y')\right)^{\alpha_1} - (F_Y(y) \wedge F_Y(y'))^{\alpha_1}\right| dydy' \\
&\quad \times \left|\left(\bar{\widehat{F}}_Y^{(1)}(y) \wedge \bar{\widehat{F}}_Y^{(2)}(y')\right)^{\alpha_2} - \left(\bar{F}_Y(y) \wedge \bar{F}_Y(y')\right)^{\alpha_2}\right| dydy' \\
\leq &\int_{\mathbb{R}^2} E^{1/2}\left[\left|\left(\widehat{F}_Y^{(1)}(y) \wedge \widehat{F}_Y^{(2)}(y')\right)^{\alpha_1} - (F_Y(y) \wedge F_Y(y'))^{\alpha_1}\right|^2\right] \\
&\quad \times E^{1/2}\left[\left|\left(\bar{\widehat{F}}_Y^{(1)}(y) \wedge \bar{\widehat{F}}_Y^{(2)}(y')\right)^{\alpha_2} - \left(\bar{F}_Y(y) \wedge \bar{F}_Y(y')\right)^{\alpha_2}\right|^2\right] dydy'. \qquad \text{(S.2.2)}
\end{aligned}$$

Now, by Lemma S.3.3 and Jensens's inequality, we have

$$\begin{aligned}
&E^{1/2}\left[\left|\left(\widehat{F}_Y^{(1)}(y) \wedge \widehat{F}_Y^{(2)}(y')\right)^{\alpha_1} - (F_Y(y) \wedge F_Y(y'))^{\alpha_1}\right|^2\right] \\
&\leq E^{\alpha_1/2}\left[\left|\widehat{F}_Y^{(1)}(y) - F_Y(y)\right|^2 \wedge F_Y(y')^2\right] + E^{\alpha_1/2}\left[\left|\widehat{F}_Y^{(2)}(y') - F_Y(y')\right|^2 \wedge F_Y(y)^2\right] \\
&\quad + E^{\alpha_1/2}\left[\left|\widehat{F}_Y^{(1)}(y) - F_Y(y)\right|^2 \wedge \left|\widehat{F}_Y^{(2)}(y') - F_Y(y')\right|^2\right] \\
&\leq \left(\frac{2F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{\alpha_1/2} \wedge F_Y(y')^{\alpha_1} + \left(\frac{2F_Y(y')\bar{F}_Y(y')}{n_1}\right)^{\alpha_1/2} \wedge F_Y(y)^{\alpha_1} \\
&\quad + \left(\frac{2F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{\alpha_1/2} \wedge \left(\frac{2F_Y(y')\bar{F}_Y(y')}{n_1}\right)^{\alpha_1/2},
\end{aligned}$$

where, in the last inequality, we used that, for any two random variables U, V, it holds that $E(U \wedge V) \leq E(U) \wedge E(V)$. The second factor can be controlled analogously. Therefore, by (S.2.2), we have

$$
\begin{aligned}
B \lesssim \int_{\mathbb{R}^2} \Bigg\{ & \left(\frac{F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{\alpha_1/2} \wedge F_Y(y')^{\alpha_1} + \left(\frac{F_Y(y')\bar{F}_Y(y')}{n_1}\right)^{\alpha_1/2} \wedge F_Y(y)^{\alpha_1} \\
&+ \left(\frac{F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{\alpha_1/2} \wedge \left(\frac{F_Y(y')\bar{F}_Y(y')}{n_1}\right)^{\alpha_1/2} \Bigg\} \\
\times \Bigg\{ & \left(\frac{F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{\alpha_2/2} \wedge \bar{F}_Y(y')^{\alpha_2} + \left(\frac{F_Y(y')\bar{F}_Y(y')}{n_1}\right)^{\alpha_2/2} \wedge \bar{F}_Y(y)^{\alpha_2} \\
&+ \left(\frac{F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{\alpha_2/2} \wedge \left(\frac{F_Y(y')\bar{F}_Y(y')}{n_1}\right)^{\alpha_2/2} \Bigg\} dydy' \\
=: 6 \int_{\mathbb{R}^2} & \Big\{ M_1(y,y') + M_2(y,y') + M_3(y,y') \Big\} \Big\{ N_1(y,y') + N_2(y,y') + N_3(y,y') \Big\} dydy'.
\end{aligned}
$$

We now prove that $\int_{\mathbb{R}^2} M_i(y,y') N_j(y,y')\, dydy' = o(1)$ for any $i, j \in \{1,2,3\}$. Using that $x \wedge y \leq (xy)^{1/2}$ for all $x, y \in [0,1]$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2} M_1(y,y') N_1(y,y')\, dy\, dy' \\
&= \int_{\mathbb{R}^2} M_2(y,y') N_2(y,y')\, dy\, dy' \\
&= \int_{\mathbb{R}^2} \left[\left(\frac{F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{\alpha_1/2} \wedge F_Y(y')^{\alpha_1} \right] \times \left[\left(\frac{F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{\alpha_2/2} \wedge \bar{F}_Y(y')^{\alpha_2} \right] dydy' \\
&\leq \int_{\mathbb{R}^2} \left(\frac{F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{(\alpha_1+\alpha_2)/2} F_Y(y')^{\alpha_1/2} \bar{F}_Y(y')^{\alpha_2/2} dydy' \\
&= o(1),
\end{aligned}
$$

where the convergence follows from Assumption 4(iv) and Lemma 6. Next,

$$
\begin{aligned}
\int_{\mathbb{R}^2} M_1(y,y') N_3(y,y') dydy' &= \int_{\mathbb{R}^2} M_2(y,y') N_3(y,y') dydy' \\
&\leq \int_{\mathbb{R}^2} \left(\frac{F_Y(y')\bar{F}_Y(y')}{n_1}\right)^{\alpha_1/2} \left(\frac{F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{\alpha_2/2} dydy' \\
&= o(1),
\end{aligned}
$$

where, again, the convergence follows from Assumption 4(iv) and Lemma 6. The terms $\int_{\mathbb{R}^2} M_2(y,y') N_1(y,y')\, dy\, dy'$ $\int_{\mathbb{R}^2} M_1(y,y') N_2(y,y')\, dy\, dy'$, $\int_{\mathbb{R}^2} M_3(y,y') N_1(y,y')\, dy\, dy' = \int_{\mathbb{R}^2} M_3(y,y') N_2(y,y')\, dy\, dy'$ and $\int_{\mathbb{R}^2} M_3(y,y') N_3(y,y')$ can be controlled analogously, which concludes the proof of the claim $B = o(1)$.

Third, we show that $J_3 = o(1)$. This follows from $\left|\widetilde{F}_Y^{(1)}(y) - F_Y(y)\right|^{\beta} \leq 1$, $g \leq C_U^2$, and Lemma 9.

Fourth, we show that $J_4 = o(1)$. Notice that

$$
\sup_{y \in \mathbb{R}} \left(h^{\beta}_{\widehat{F}_Y^{(1)}(y)} + h^{\beta}_{\widehat{\widehat{F}}_Y^{(1)}(y)} + h^{\beta}_{\widehat{F}_Y^{(2)}(y')} + h^{\beta}_{\widehat{\widehat{F}}_Y^{(2)}(y')} \right) \leq 4\varepsilon_{n_2} = o(1).
$$

Hence,

$$\begin{aligned}
J_4 &\le 64c_U^2\varepsilon_{n_2}^2 E\left[\left(\int_{\mathbb{R}^2}\Omega\left(\widetilde{F}_Y^{(1)}(y),\widetilde{F}_Y^{(2)}(y')\right)dydy'\right)^2\right]\\
&\quad + 4E\left[\left(\int_{\mathbb{R}^2}\left(\left|\widehat{F}_Y^{(1)}(y)-\widehat{\widehat{F}}_Y^{(1)}(y)\right|^\beta+\left|\widehat{F}_Y^{(2)}(y')-\widehat{\widehat{F}}_Y^{(2)}(y')\right|^\beta\right)\Omega\left(\widetilde{F}_Y^{(1)}(y),\widetilde{F}_Y^{(2)}(y')\right)dydy'\right)^2\right]\\
&= o(1) + 4E\left[\left(\int_{\mathbb{R}^2}\left(\left|\widehat{F}_Y^{(1)}(y)-\widehat{\widehat{F}}_Y^{(1)}(y)\right|^\beta+\left|\widehat{F}_Y^{(2)}(y')-\widehat{\widehat{F}}_Y^{(2)}(y')\right|^\beta\right)\Omega\left(\widetilde{F}_Y^{(1)}(y),\widetilde{F}_Y^{(2)}(y')\right)dydy'\right)^2\right],
\end{aligned}$$

where the last equality follows from

$$E\left[\left(\int_{\mathbb{R}^2}\Omega\left(\widetilde{F}_Y^{(1)}(y),\widetilde{F}_Y^{(2)}(y')\right)dydy'\right)^2\right] = (1+O(\varepsilon_{n_2}))^2\, E\left[\left(\int_{\mathbb{R}^2}\Omega\left(\widehat{F}_Y^{(1)}(y),\widehat{F}_Y^{(2)}(y')\right)dydy'\right)^2\right]$$

which is finite by Lemma 9 since $1-2b_j > 2d_j$ for $j\in\{1,2\}$ by Assumption 2(iv). The remaining term can be shown to converge to zero by adding and subtracting $F_Y(y)$ and $F_Y(y')$ inside each absolute value, respectively, by using the sub-additivity of $x\mapsto |x|^\beta$, and by following a reasoning similar to Step 1.A above for showing that $J_1 = o(1)$ as $N\to\infty$.

Fifth, that $J_5 = o(1)$ follows by adding and subtracting $\Omega(F_Y(y),F_Y(y'))$ inside the absolute value and by following a reasoning similar to the reasoning above for showing that $J_2 = o(1)$ as $N\to\infty$. □

Lemma S.2.2 *For $k\in\{6,7,8\}$: $E[|J_k|] = o(1)$ as $N\to\infty$.*

Proof: First, we show that $E[|J_6|] = o(1)$. Using symmetry, we have the decomposition

$$\begin{aligned}
&E[|J_6|]\\
\le& E\left[\int_{[0,1]^4}\operatorname{cov}(\widehat{f}_U^{(1)}(s),\widehat{f}_U^{(1)}(s'))\operatorname{cov}(\widehat{f}_U^{(1)}(t),\widehat{f}_U^{(1)}(t'))d\widehat{\mu}_{\mathbf{Y}}(s,t,s',t')\right]\\
&+E\left[\left(\int_{[0,1]^2}\operatorname{cov}(\widehat{f}_U^{(1)}(s),\widehat{f}_U^{(1)}(t))d\widehat{\mu}_{\mathbf{Y}}(s,t)\right)^2\right]\\
&+2E\left[\int_{[0,1]^4}E(\widehat{f}_U^{(1)}(s))E(\widehat{f}_U^{(1)}(s'))\operatorname{cov}(\widehat{f}_U^{(1)}(t),\widehat{f}_U^{(1)}(t'))d\widehat{\mu}_{\mathbf{Y}}(s,t,s',t')\right]\\
&+2E\left[\int_{[0,1]^4}E(\widehat{f}_U^{(1)}(s))E(\widehat{f}_U^{(1)}(t))\operatorname{cov}(\widehat{f}_U^{(1)}(s'),\widehat{f}_U^{(1)}(t'))d\widehat{\mu}_{\mathbf{Y}}(s,t,s',t')\right]\\
=&o(1),
\end{aligned}$$

where the convergence follows by Lemma S.2.3.

Second, we show that $E\left[|J_7|+|J_8|\right] = o(1)$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&E\left[|J_7|+|J_8|\right]\\
&\le E\left[\left(\int_{[0,1]^2}\operatorname{cov}(\widehat{f}_U^{(1)}(s),\widehat{f}_U^{(1)}(t))d\widehat{\mu}_{\mathbf{Y}}(s,t)\right)^2\right]
\end{aligned}$$

$$+ E^{1/2}\left[\left(\int_{[0,1]^2} \mathrm{cov}(\widehat{f}_U^{(1)}(s), \widehat{f}_U^{(1)}(t))\, d\widehat{\mu}_{\mathbf{Y}}(s,t)\right)^2\right] E^{1/2}\left[\left(\int_{[0,1]^2} e_{\mathbf{Y}}(s) e_{\mathbf{Y}}(t)\, d\widehat{\mu}_{\mathbf{Y}}(s,t)\right)^2\right].$$

By Lemma S.2.3(i) below, we have $E\left[\left(\int_{[0,1]^2} \mathrm{cov}(\widehat{f}_U^{(1)}(s), \widehat{f}_U^{(1)}(t)) d\widehat{\mu}_{\mathbf{Y}}(s,t)\right)^2\right] = o(1)$. To show that $E\left[|J_7| + |J_8|\right] = o(1)$, it now remains to prove that $E\left[\left(\int_{[0,1]^2} e_{\mathbf{Y}}(s) e_{\mathbf{Y}}(t) d\widehat{\mu}_{\mathbf{Y}}(s,t)\right)^2\right] = O(1)$. We recall the definition of $\widetilde{f}_U^{(1)}$ from (29) and will slightly abuse notation by writing h_s instead of $h_{n_2,s}$. We have, for any $s \in [0,1]$,

$$\begin{aligned}
e_{\mathbf{Y}}(s) = E\left[\widetilde{f}_U^{(1)}(s) \mid \mathbf{Y}\right] &= \frac{1}{2h_s}\Pr\left(|U - s| \le h_s\right) \\
&= \frac{F_U(s + h_s) - F_U(s - h_s)}{2h_s} \\
&= f_U(\tilde{s}) \\
&\le \frac{\widetilde{C}_U}{s^{b_1}\bar{s}^{b_2}},
\end{aligned}$$

for some absolute constant $\widetilde{C}_U > 0$ where we used the mean value theorem for some $\tilde{s} \in [s - h_s, s + h_s]$ in the third line, and Lemma S.2.4 in the last line. We therefore obtain

$$\begin{aligned}
&E\left[\left(\int_{[0,1]^2} e_{\mathbf{Y}}(s)\, e_{\mathbf{Y}}(t)\, d\widehat{\mu}_{\mathbf{Y}}(s,t)\right)^2\right] \\
&\le E\left[\left(\int_{[0,1]^2} \frac{\widetilde{C}_U^2}{(st)^{b_1}(\bar{s}\bar{t})^{b_2}}(s \wedge t)(\bar{s} \wedge \bar{t})\, d(\widehat{F}_Y^{(1)})^{-1}(s)\, d(\widehat{F}_Y^{(2)})^{-1}(t)\right)^2\right] \\
&\le E\left[\left(\int_{[0,1]^2} \frac{\widetilde{C}_U^2}{(s \wedge t)^{2b_1}(\bar{s} \wedge \bar{t})^{2b_2}}(s \wedge t)(\bar{s} \wedge \bar{t})\, d(\widehat{F}_Y^{(1)})^{-1}(s)\, d(\widehat{F}_Y^{(2)})^{-1}(t)\right)^2\right] \\
&= \widetilde{C}_U^4 E\left[\left(\int_{[0,1]^2} (s \wedge t)^{1-2b_1}(\bar{s} \wedge \bar{t})^{1-2b_2}\, d(\widehat{F}_Y^{(1)})^{-1}(s)\, d(\widehat{F}_Y^{(2)})^{-1}(t)\right)^2\right],
\end{aligned}$$

which is finite by Lemma 9 since $1 - 2b_j > 2d_j$ for $j \in \{1,2\}$ by Assumption 2(iv). □

Lemma S.2.3 *It holds that*

(i) $E\left[\left(\int_{[0,1]^2} \mathrm{cov}(\widehat{f}_U^{(1)}(s), \widehat{f}_U^{(1)}(t)) d\widehat{\mu}_{\mathbf{Y}}(s,t)\right)^2\right] = o(1)$.

(ii) $E\left[\int_{[0,1]^4} \mathrm{cov}(\widehat{f}_U^{(1)}(s), \widehat{f}_U^{(1)}(s'))\mathrm{cov}(\widehat{f}_U^{(1)}(t), \widehat{f}_U^{(1)}(t')) d\widehat{\mu}_{\mathbf{Y}}(s,t,s',t')\right] = o(1)$.

(iii) $E\left[\int_{[0,1]^4} E(\widehat{f}_U^{(1)}(s))E(\widehat{f}_U^{(1)}(s'))\mathrm{cov}(\widehat{f}_U^{(1)}(t), \widehat{f}_U^{(1)}(t')) d\widehat{\mu}_{\mathbf{Y}}(s,t,s',t')\right] = o(1)$.

(iv) $E\left[\int_{[0,1]^4} E(\widehat{f}_U^{(1)}(s))E(\widehat{f}_U^{(1)}(t))\mathrm{cov}(\widehat{f}_U^{(1)}(s'), \widehat{f}_U^{(1)}(t')) d\widehat{\mu}_{\mathbf{Y}}(s,t,s',t')\right] = o(1)$.

Proof: For ease of notation, we will write n and $\widehat{F}^{(j)}$ instead of n_2 and $\widehat{F}_Y^{(j)}$ respectively throughout the proof. We have, for any $s, t \in (0,1)$,

$$\begin{aligned}
&\operatorname{cov}(\widehat{f}_U^{(1)}(s), \widehat{f}_U^{(1)}(t)) \\
&\quad= E\left[\frac{1}{n^2 h_s h_t}\left(\sum_{i=1}^{n/2}\mathbb{1}_{|U_i-s|\le h_s}\right)\left(\sum_{j=1}^{n/2}\mathbb{1}_{|U_j-t|\le h_t}\right)\right] - \frac{1}{n^2 h_s h_t}E\left[\sum_{i=1}^{n/2}\mathbb{1}_{|U_i-s|\le h_s}\right]E\left[\sum_{j=1}^{n/2}\mathbb{1}_{|U_j-t|\le h_t}\right] \\
&\quad= \frac{1}{n^2 h_s h_t}E\left[\sum_{i\ne j}\mathbb{1}_{|U_i-s|\le h_s}\mathbb{1}_{|U_j-t|\le h_t} + \sum_{i=1}^{n}\mathbb{1}_{|U_i-s|\le h_s}\mathbb{1}_{|U_i-t|\le h_t}\right] \\
&\qquad - \frac{1}{h_s h_t}\Pr\left(|U-s|\le h_s\right)\Pr\left(|U-t|\le h_t\right) \\
&\quad\le \frac{1}{h_s h_t}\Pr\left(|U-s|\le h_s\right)\Pr\left(|U-t|\le h_t\right) + \frac{1}{n h_s h_t}\Pr\left(|U-s|\le h_s \text{ and } |U-t|\le h_t\right) \\
&\qquad - \frac{1}{h_s h_t}\Pr\left(|U-s|\le h_s\right)\Pr\left(|U-t|\le h_t\right) \\
&\quad= \frac{1}{n h_s h_t}\Pr\left(|U-s|\le h_s \text{ and } |U-t|\le h_t\right).
\end{aligned}$$

Moreover, by Lemma S.3.7 below, we have that $|s-t| \le 7h_s$ and $h_t \ge h_s/6$ whenever $|s-t| \le h_s + h_t$. Therefore,

$$\begin{aligned}
\frac{1}{n h_s h_t}\Pr\left(|U-s|\le h_s \text{ and } |U-t|\le h_t\right) &\le \frac{1}{n h_s h_t}\Pr\left(|U-s|\le h_s \text{ and } |s-t|\le h_s+h_t\right) \\
&= \frac{1}{n h_s h_t}\Pr\left(|U-s|\le h_s\right)\mathbf{1}\left\{|s-t|\le h_s+h_t\right\} \\
&\le \frac{1}{n h_s\cdot(h_s/6)}\cdot 2h_s f_U(\tilde{s})\mathbb{1}\{|s-t|\le 7h_s\} \\
&\le \frac{12}{n h_s}\frac{\widetilde{C}_U}{s^{b_1}\bar{s}^{b_2}}\mathbb{1}\{|s-t|\le 7h_s\} \\
&=: \frac{C}{n h_s}\frac{1}{s^{b_1}\bar{s}^{b_2}}\mathbb{1}\{|s-t|\le 7h_s\}.
\end{aligned}$$

Therefore,

$$\operatorname{cov}(\widehat{f}_U^{(1)}(s),\ \widehat{f}_U^{(1)}(t)) \le \frac{C}{n h_s}\frac{1}{s^{b_1}\bar{s}^{b_2}}\mathbb{1}\{|s-t|\le 7h_s\}.$$

Whenever $|s-t| \le 7h_s$, we have $t \le s + 7h_s \le 5s$ since $h_s \le \frac{1}{2}s\bar{s}$, and similarly, $\bar{t} \le 5\bar{s}$. Letting $n_1' = n_1/2$, we have

$$\begin{aligned}
&E\left[\left(\int_{[0,1]^2}\operatorname{cov}(\widehat{f}_U^{(1)}(s), \widehat{f}_U^{(1)}(t))d\widehat{\mu}_{\mathbf{Y}}(s,t)\right)^2\right] \\
\le &E\left[\left(\int_{[0,1]^2}\frac{C}{n h_s}\frac{1}{s^{b_1}\bar{s}^{b_2}}\mathbb{1}\{|s-t|\le 7h_s\}d\widehat{\mu}_{\mathbf{Y}}(s,t)\right)^2\right]
\end{aligned}$$

$$
\begin{aligned}
&= E\left[\left(\int_{[0,1]^2} \frac{C}{nh_s}\frac{1}{s^{b_1}\bar{s}^{b_2}}\mathbb{1}\{|s-t|\le 7h_s\}(s\wedge t)(\bar{s}\wedge\bar{t})\, d(\widehat{F}^{(1)})^{-1}(s)\, d(\widehat{F}^{(2)})^{-1}(t)\right)^2\right] \\
&\le E\left[\left(\int_{[0,1]^2} \frac{C}{nh_s}\frac{1}{s^{b_1}\bar{s}^{b_2}}\mathbb{1}\{|s-t|\le 7h_s\}\cdot 5s\cdot 5\bar{s}\cdot\, d(\widehat{F}^{(1)})^{-1}(s)\, d(\widehat{F}^{(2)})^{-1}(t)\right)^2\right] \\
&= (25C)^2 E\left[\left(\frac{1}{n}\int_{[0,1]} s^{1-b_1}\bar{s}^{1-b_2}\left(\frac{1}{h_s}\int_{[0,1]}\mathbb{1}\{|s-t|\le 7h_s\} d(\widehat{F}^{(2)})^{-1}(t)\right) d(\widehat{F}^{(1)})^{-1}(s)\right)^2\right] \\
&\lesssim E\left[\left(\frac{1}{n}\int_{[0,1]} s^{1-b_1}\bar{s}^{1-b_2}\frac{(\widehat{F}^{(2)})^{-1}(s+7h_s)-(\widehat{F}^{(2)})^{-1}(s-7h_s)}{h_s} d(\widehat{F}^{(1)})^{-1}(s)\right)^2\right] \\
&= E\left[\left(\frac{1}{n}\int_{[0,1]} s^{1-b_1}\bar{s}^{1-b_2}\frac{Y^{(2)}_{(\lceil n_1's+7n_1'h_s\rceil)}-Y^{(2)}_{(\lceil n_1's-7n_1'h_s\rceil)}}{h_s} d(\widehat{F}^{(1)})^{-1}(s)\right)^2\right] \\
&= E\left[\left(\frac{1}{n}\sum_{k=1}^{n_1'-1}\left(\frac{k}{n_1'}\right)^{1-b_1}\left(\frac{n_1'-k}{n_1'}\right)^{1-b_2}\frac{Y^{(2)}_{(\lceil k+7n_1'h_{k/n_1'}\rceil)}-Y^{(2)}_{(\lceil k-7n_1'h_{k/n_1'}\rceil)}}{h_{k/n_1'}}\left(Y^{(1)}_{(k+1)}-Y^{(1)}_{(k)}\right)\right)^2\right] \\
&= \frac{1}{n^2\varepsilon_n^2}\sum_{k,\ell=1}^{n_1'-1}\left(\frac{k\ell}{(n_1')^2}\right)^{-b_1}\left(\frac{(n_1'-k)(n_1'-\ell)}{(n_1')^2}\right)^{-b_2} E\Big[\left(Y^{(2)}_{(\lceil k+7n_1'h_{k/n_1'}\rceil)}-Y^{(2)}_{(\lceil k-7n_1'h_{k/n_1'}\rceil)}\right)\left(Y^{(1)}_{(k+1)}-Y^{(1)}_{(k)}\right) \\
&\qquad\qquad \times\left(Y^{(2)}_{(\lceil \ell+7n_1'h_{\ell/n_1'}\rceil)}-Y^{(2)}_{(\lceil \ell-7n_1'h_{\ell/n_1'}\rceil)}\right)\left(Y^{(1)}_{(\ell+1)}-Y^{(1)}_{(\ell)}\right)\Big] \\
&\le \frac{1}{n^2\varepsilon_n^2}\sum_{k,\ell=1}^{n_1'-1}\left(\frac{k\ell}{(n_1')^2}\right)^{-b_1}\left(\frac{(n_1'-k)(n_1'-\ell)}{(n_1')^2}\right)^{-b_2} E\Big[\left(Y^{(2)}_{(\lceil k+7n_1'h_{k/n_1'}\rceil)}-Y^{(2)}_{(\lceil k-7n_1'h_{k/n_1'}\rceil)}\right)^2 \\
&\qquad\qquad \times\left(Y^{(1)}_{(\lceil \ell+7n_1'h_{\ell/n_1'}\rceil)}-Y^{(1)}_{(\lceil \ell-7n_1'h_{\ell/n_1'}\rceil)}\right)^2\Big] \\
&= \left(\frac{1}{n\varepsilon_n}\sum_{k=1}^{n_1'-1}\left(\frac{k}{n_1'}\right)^{-b_1}\left(\frac{(n_1'-k)}{n_1'}\right)^{-b_2} E\Big[\left(Y^{(1)}_{(\lceil k+7n_1'h_{k/n_1'}\rceil)}-Y^{(1)}_{(\lceil k-7n_1'h_{k/n_1'}\rceil)}\right)^2\Big]\right)^2 \\
&\lesssim \Bigg(\frac{1}{n\varepsilon_n}\sum_{k=1}^{n_1'-1}\left(\frac{k}{n_1'}\right)^{-b_1}\left(\frac{n_1'-k}{n_1'}\right)^{-b_2}\left(\frac{h_{k/n_1'}+\frac{1}{n_1'}}{\left(\frac{k}{n_1'}\right)^{1+d_1}\left(\frac{n_1'-k}{n_1'}\right)^{1+d_2}}\right)^2\Bigg)^2,
\end{aligned}
$$

where we used the independence between $(Y_i^{(1)})_i$ and $(Y_j^{(2)})_j$ to obtain the second to last line, and Lemma S.3.1 below in the last inequality.
Next, since $n_1 \asymp n_2$, we have

$$
\frac{1}{n\varepsilon_n}\sum_{k=1}^{n_1'-1}\left(\frac{k}{n_1'}\right)^{-b_1}\left(\frac{n_1'-k}{n_1'}\right)^{-b_2}\left(\frac{h_{k/n_1'}+\frac{1}{n_1'}}{\left(\frac{k}{n_1'}\right)^{1+d_1}\left(\frac{n_1'-k}{n_1'}\right)^{1+d_2}}\right)^2
$$

$$
\begin{aligned}
&\leq \frac{2}{n\varepsilon_n}\sum_{k=1}^{n_1'-1}\left(\frac{k}{n_1'}\right)^{-b_1-2(1+d_1)}\left(\frac{n_1'-k}{n_1'}\right)^{-b_2-2(1+d_2)}\left(h^2_{\frac{k}{n_1'}}+\frac{1}{(n_1')^2}\right)\\
&\asymp \frac{2\varepsilon_n}{n}\sum_{k=1}^{n_1'-1}\left(\frac{k}{n_1'}\right)^{-b_1-2d_1}\left(\frac{n_1'-k}{n_1'}\right)^{-b_2-2d_2}+\frac{2}{n^3\varepsilon_n}\sum_{k=1}^{n_1'-1}\left(\frac{k}{n_1'}\right)^{-b_1-2(1+d_1)}\left(\frac{n_1'-k}{n_1'}\right)^{-b_2-2(1+d_2)}\\
&\lesssim \frac{2\varepsilon_n}{n}\sum_{k=1}^{n_1'-1}\left[\left(\frac{k}{n_1'}\right)^{-b_1-2d_1}+\left(\frac{n_1'-k}{n_1'}\right)^{-b_2-2d_2}\right]+\frac{2}{n^3\varepsilon_n}\sum_{k=1}^{n_1'-1}\left[\left(\frac{k}{n_1'}\right)^{-b_1-2(1+d_1)}+\left(\frac{n_1'-k}{n_1'}\right)^{-b_2-2(1+d_2)}\right].
\end{aligned}
$$

In the last step, we used the relation $x^u(1-x)^v \lesssim x^u+(1-x)^v$ for $u,v\in\mathbb{R}$ and any $x\in(0,1)$. Fix $j\in\{1,2\}$. Under Assumption 4(iv), we have

$$
\frac{2\varepsilon_n}{n}\sum_{k=1}^{n_1'-1}\left(\frac{k}{n_1'}\right)^{-b_j-2d_j}\lesssim \varepsilon_n=o(1),
$$

and

$$
\frac{2}{n^3\varepsilon_n}\sum_{k=1}^{n_1'-1}\left(\frac{k}{n_1'}\right)^{-b_j-2(1+d_j)}\lesssim \frac{n^{b_j+2(1+d_j)}}{n^3\varepsilon_n}=o(1),
$$

which implies that

$$
\frac{1}{n\varepsilon_n}\sum_{k=1}^{n_1'-1}\left(\frac{k}{n_1'}\right)^{-b_1}\left(\frac{n_1'-k}{n_1'}\right)^{-b_2}\left(\frac{h_{k/n_1'}+\frac{1}{n_1'}}{\left(\frac{k}{n_1'}\right)^{1+d_1}\left(\frac{n_1'-k}{n_1'}\right)^{1+d_2}}\right)^2=o(1).
$$

This completes the proof of Point 1.
As for Point 2, an analogous reasoning yields

$$
\begin{aligned}
&E\left[\int_{[0,1]^4}\operatorname{cov}(\widehat{f}_U^{(1)}(s),\widehat{f}_U^{(1)}(s'))\operatorname{cov}(\widehat{f}_U^{(1)}(t),\widehat{f}_U^{(1)}(t'))d\widehat{\mu}_{\mathbf{Y}}(s,t,s',t')\right]\\
\leq &E\left[\int_{[0,1]^4}\frac{C^2}{n^2h_s}\frac{s\bar{s}}{s^{b_1}\bar{s}^{b_2}}\mathbb{1}\{|s-s'|\leq 7h_s\}\frac{1}{h_{t'}}\frac{t'\bar{t}'}{t'^{b_1}\bar{t}'^{b_2}}\mathbb{1}\{|t'-t|\leq 7h_{t'}\}d\widehat{\mu}_{\mathbf{Y}}(s,t,s',t')\right]\\
=&C^2E\Bigg[\left(\frac{1}{n}\int_{[0,1]}s^{1-b_1}\bar{s}^{1-b_2}\frac{Y^{(1)}_{(\lceil n_1's+7n_1'h_s\rceil)}-Y^{(1)}_{(\lceil n_1's-7n_1'h_s\rceil)}}{h_s}d(\widehat{F}^{(1)})^{-1}(s)\right)\\
&\times\left(\frac{1}{n}\int_{[0,1]}t'^{1-b_1}\bar{t}'^{1-b_2}\frac{Y^{(2)}_{(\lceil n_1't'+7n_1'h_{t'}\rceil)}-Y^{(2)}_{(\lceil n_1't'-7n_1'h_{t'}\rceil)}}{h_{t'}}d(\widehat{F}^{(2)})^{-1}(t')\right)\Bigg]\\
=&C^2E\left[\frac{1}{n}\int_{[0,1]}s^{1-b_1}\bar{s}^{1-b_2}\frac{Y^{(1)}_{(\lceil n_1's+7n_1'h_s\rceil)}-Y^{(1)}_{(\lceil n_1's-7n_1'h_s\rceil)}}{h_s}d(\widehat{F}^{(1)})^{-1}(s)\right]\\
&\times C^2E\left[\frac{1}{n}\int_{[0,1]}t'^{1-b_1}\bar{t}'^{1-b_2}\frac{Y^{(2)}_{(\lceil n_1't'+7n_1'h_{t'}\rceil)}-Y^{(2)}_{(\lceil n_1't'-7n_1'h_{t'}\rceil)}}{h_{t'}}d(\widehat{F}^{(2)})^{-1}(t')\right],
\end{aligned}
$$

where the last equality follows from independence between the samples $(Y_i^{(1)})_i$ and $(Y_i^{(2)})_i$. That each expectation in the upper bound is $o(1)$ follows from the Cauchy–Schwarz inequality together with the same arguments used in the proof of Point 1.
As for Point 3, we have

$$
\begin{aligned}
&E\left[\int_{[0,1]^4} E(\widehat{f}_U^{(1)}(s))E(\widehat{f}_U^{(1)}(s'))\mathrm{cov}(\widehat{f}_U^{(1)}(t),\widehat{f}_U^{(1)}(t'))d\widehat{\mu}_{\mathbf{Y}}(s,t,s',t')\right]\\
\leq& C\widetilde{C}_U^2 E\Bigg[\left(\int_{[0,1]} s^{1-b_1}\bar{s}^{1-b_2}d(\widehat{F}^{(1)})^{-1}(s)\right)\left(\int_{[0,1]} {s'}^{1/2-b_1}\bar{s'}^{1/2-b_2}d(\widehat{F}^{(1)})^{-1}(s')\right)\\
&\quad\times\left(\frac{1}{n}\int_{[0,1]} {t'}^{1/2}\bar{t'}^{1/2}\frac{Y^{(2)}_{(\lceil n_1't'+7n_1'h_{t'}\rceil)}-Y^{(2)}_{(\lceil n_1't'-7n_1'h_{t'}\rceil)}}{h_{t'}}d(\widehat{F}^{(2)})^{-1}(t')\right)\Bigg]\\
\leq& C\widetilde{C}_U^2 E^{1/2}\left[\left(\int_{[0,1]} s^{1-b_1}\bar{s}^{1-b_2}d(\widehat{F}^{(1)})^{-1}(s)\right)^2\right]E^{1/2}\left[\left(\int_{[0,1]} {s'}^{1/2-b_1}\bar{s'}^{1/2-b_2}d(\widehat{F}^{(1)})^{-1}(s')\right)^2\right]\\
&\quad\times E\left[\left(\frac{1}{n}\int_{[0,1]} {t'}^{1/2}\bar{t'}^{1/2}\frac{Y^{(2)}_{(\lceil n_1't'+7n_1'h_{t'}\rceil)}-Y^{(2)}_{(\lceil n_1't'-7n_1'h_{t'}\rceil)}}{h_{t'}}d(\widehat{F}^{(2)})^{-1}(t')\right)\right],
\end{aligned}
$$

where the last inequality follows from the Cauchy–Schwarz inequality and independence between the samples $(Y_i^{(1)})_i$ and $(Y_i^{(2)})_i$. That the upper bound is $o(1)$ follows from the first two expectations being bounded (by Lemma 9) and the last expectation being $o(1)$ by the Cauchy–Schwarz inequality and the same arguments used in the proof of Point 1.
Finally, as for Point 4, we have

$$
\begin{aligned}
&E\left[\int_{[0,1]^4} E(\widehat{f}_U^{(1)}(s))E(\widehat{f}_U^{(1)}(t))\mathrm{cov}(\widehat{f}_U^{(1)}(s'),\widehat{f}_U^{(1)}(t'))d\widehat{\mu}_{\mathbf{Y}}(s,t,s',t')\right]\\
\leq& C\widetilde{C}_U^2 E\Bigg[\left(\int_{[0,1]} s^{1/2-b_1}\bar{s}^{1/2-b_2}d(\widehat{F}^{(1)})^{-1}(s)\right)\left(\int_{[0,1]} t^{1/2-b_1}\bar{t}^{1/2-b_2}d(\widehat{F}^{(2)})^{-1}(t)\right)\\
&\quad\times\left(\frac{1}{n}\int_{[0,1]} {s'}^{1-b_1}\bar{s'}^{1-b_2}\frac{Y^{(2)}_{(\lceil n_1's'+7n_1'h_{s'}\rceil)}-Y^{(2)}_{(\lceil n_1's'-7n_1'h_{s'}\rceil)}}{h_{s'}}d(\widehat{F}^{(1)})^{-1}(s')\right)\Bigg]\\
\leq& C\widetilde{C}_U^2 E^{1/2}\left[\left(\int_{[0,1]} s^{1/2-b_1}\bar{s}^{1/2-b_2}d(\widehat{F}^{(1)})^{-1}(s)\right)^2\right]E^{1/2}\left[\left(\int_{[0,1]} t^{1/2-b_1}\bar{t}^{1/2-b_2}d(\widehat{F}^{(2)})^{-1}(t)\right)^2\right]\\
&\quad\times E^{1/2}\left[\left(\frac{1}{n}\int_{[0,1]} {s'}^{1/2}\bar{s'}^{1/2}\frac{Y^{(2)}_{(\lceil n_1's'+7n_1'h_{s'}\rceil)}-Y^{(2)}_{(\lceil n_1's'-7n_1'h_{s'}\rceil)}}{h_{s'}}d(\widehat{F}^{(1)})^{-1}(s')\right)\right],
\end{aligned}
$$

where the last inequality follows from the Cauchy–Schwarz inequality and independence between the samples $(Y_i^{(1)})_i$ and $(Y_i^{(2)})_i$. That the upper bound is $o(1)$ follows from the first two expectations being bounded (by Lemma 9) and the last expectation being $o(1)$ by the Cauchy–Schwarz inequality and the same arguments used in the proof of Point 1.

□

Lemma S.2.4 *There exists some absolute constant* $\widetilde{C}_U > 0$ *such that, for all* $s, \tilde{s} \in (0,1)$ *such that* $\tilde{s} \in [s - h_s, s + h_s]$,

$$f_U(\tilde{s}) \leq \frac{\widetilde{C}_U}{s^{b_1}\bar{s}^{b_2}}.$$

Proof: Without loss of generality, assume that $s \in (0, 1/2]$ (the symmetric case $s \in [1/2, 1)$ can be treated analogously). By Assumption 2(iii), we have

$$f_U(\tilde{s}) \leq \frac{C_U}{\tilde{s}^{b_1}\bar{\tilde{s}}^{b_2}}.$$

If $s \in [1/4, 1/2]$ then $\tilde{s} \in [1/8, 3/4]$ since we have $h_s \leq \frac{1}{2}s\bar{s}$ by definition. We then obtain

$$f_U(\tilde{s}) \leq C \leq \frac{C'}{s^{b_1}\bar{s}^{b_2}}$$

for some absolute constant $C' > 0$. Otherwise, $s \in (0, 1/4]$ so that $\tilde{s} \leq 1/2$ and $\tilde{s} \geq s/2$. We therefore obtain

$$f_U(\tilde{s}) \leq \frac{C_U}{\tilde{s}^{b_1}\bar{\tilde{s}}^{b_2}} \leq \frac{C_U}{(s/2)^{b_1}(1/2)^{b_2}} \leq \frac{2^{b_1+b_2}C_U}{s^{b_1}\bar{s}^{b_2}} \leq \frac{2C_U}{s^{b_1}\bar{s}^{b_2}}.$$

□

Lemma S.2.5

$$E\left[\int_{\mathbb{R}^2} \frac{w(\widehat{F}_Y^{(1)}(y), \widehat{F}_Y^{(2)}(y'))}{h_{\widehat{F}_Y^{(1)}(y)} h_{\widehat{F}_Y^{(2)}(y')}} I(y,y')dydy'\right] = o(1), \tag{S.2.3}$$

$$E\left[\int_{\mathbb{R}^2} \frac{w(\widehat{F}_Y^{(1)}(y), \widehat{F}_Y^{(2)}(y'))}{h_{\widehat{F}_Y^{(1)}(y)} h_{\widehat{F}_Y^{(2)}(y')}} II(y,y')dydy'\right] = o(1), \tag{S.2.4}$$

$$E\left[\int_{\mathbb{R}^2} \frac{w(\widehat{F}_Y^{(1)}(y), \widehat{F}_Y^{(2)}(y'))}{h_{\widehat{F}_Y^{(1)}(y)} h_{\widehat{F}_Y^{(2)}(y')}} III(y,y')dydy'\right] = o(1). \tag{S.2.5}$$

Proof: First, we show (S.2.3). By independence of U_2 from $(Z_1, \ldots, Z_{n_3}, Y_1, \ldots, Y_{n_1}, X_1)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{w(\widehat{F}_Y^{(1)}(y), \widehat{F}_Y^{(2)}(y'))}{h_{\widehat{F}_Y^{(1)}(y)} h_{\widehat{F}_Y^{(2)}(y')}} I(y,y')dydy' \\
&= \int_{\mathbb{R}^2} \frac{w(\widehat{F}_Y^{(1)}(y), \widehat{F}_Y^{(2)}(y'))}{h_{\widehat{F}_Y^{(1)}(y)} h_{\widehat{F}_Y^{(2)}(y')}} \\
&\qquad \times E_{\mathbf{Y}}\Big[\Big|\mathbb{1}\left\{\left|\widehat{U}^{(1)} - \widehat{F}_Y^{(1)}(y)\right| \leq h_{\widehat{F}_Y^{(1)}(y)}\right\} - \mathbb{1}\left\{\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \leq h_{\widehat{F}_Y^{(1)}(y)}\right\}\Big| \times \mathbb{1}\{\mathcal{A}_n^{(1)}\}\Big] \\
&\qquad \times \mathrm{Pr}_{\mathbf{Y}}\left(\left|U_2 - \widehat{F}_Y^{(2)}(y')\right| \leq h_{\widehat{F}_Y^{(2)}(y')}\right) dydy' \\
&= \int_{\mathbb{R}^2} \frac{w(\widehat{F}_Y^{(1)}(y), \widehat{F}_Y^{(2)}(y'))}{h_{\widehat{F}_Y^{(1)}(y)}}
\end{aligned}$$

$$\times E_{\mathbf{Y}}\Big[\Big|\mathbb{1}\Big\{\Big|\widehat{U}^{(1)} - \widehat{F}_Y^{(1)}(y)\Big| \le h_{\widehat{F}_Y^{(1)}(y)}\Big\} - \mathbb{1}\Big\{\Big|U_1 - \widehat{F}_Y^{(1)}(y)\Big| \le h_{\widehat{F}_Y^{(1)}(y)}\Big\}\Big| \times \mathbb{1}\{\mathcal{A}_n^{(1)}\}\Big]$$

$$\times f_U(\widetilde{F}_Y(y'))dydy',$$

by the mean-value theorem for some $\widetilde{F}_Y(y') \in (\widehat{F}_Y^{(2)}(y') \pm h_{\widehat{F}_Y^{(2)}(y')})$. Fix $\delta > 0$ such that $\delta < 1 - 2[(b_1 + d_1) \vee (b_2 + d_2)]$ which exists by Assumption 2(iv). We have for all $y, y' \in \mathbb{R}$

$$w(\widehat{F}_Y^{(1)}(y), \widehat{F}_Y^{(2)}(y')) \le \widehat{F}_Y^{(1)}(y)^{1-b_1-d_1-\delta}\widehat{F}_Y^{(2)}(y')^{b_1+d_1+\delta}\bar{\widehat{F}}_Y^{(1)}(y)^{1-b_2-d_2-\delta}\bar{\widehat{F}}_Y^{(2)}(y')^{b_2+d_2+\delta}.$$

Hence,

$$\int_{\mathbb{R}^2} \frac{w(\widehat{F}_Y^{(1)}(y), \widehat{F}_Y^{(2)}(y'))}{h_{\widehat{F}_Y^{(1)}(y)}h_{\widehat{F}_Y^{(2)}(y')}} I(y, y')dydy'$$

$$\lesssim \int_{\mathbb{R}^2} \frac{\widehat{F}_Y^{(1)}(y)^{1-b_1-d_1-\delta}\widehat{F}_Y^{(2)}(y')^{b_1+d_1+\delta}\bar{\widehat{F}}_Y^{(1)}(y)^{1-b_2-d_2-\delta}\bar{\widehat{F}}_Y^{(2)}(y')^{b_2+d_2+\delta}}{h_{\widehat{F}_Y^{(1)}(y)}}$$

$$\times E_{\mathbf{Y}}\Big[\Big|\mathbb{1}\Big\{\Big|\widehat{U}_1 - \widehat{F}_Y^{(1)}(y)\Big| \le h_{\widehat{F}_Y^{(1)}(y)}\Big\} - \mathbb{1}\Big\{\Big|U_1 - \widehat{F}_Y^{(1)}(y)\Big| \le h_{\widehat{F}_Y^{(1)}(y)}\Big\}\Big| \times \mathbb{1}\{\mathcal{A}_n^{(1)}\}\Big]$$

$$\times (\widetilde{F}_Y^{(2)}(y'))^{-b_1}(\bar{\widetilde{F}}_Y^{(2)}(y'))^{-b_2}dydy'$$

$$\lesssim \frac{1}{\varepsilon_n}\int_{\mathbb{R}^2} \widehat{F}_Y^{(1)}(y)^{-b_1-d_1-\delta}\widehat{F}_Y^{(2)}(y')^{d_1+\delta}\bar{\widehat{F}}_Y^{(1)}(y)^{-b_2-d_2-\delta}\bar{\widehat{F}}_Y^{(2)}(y')^{d_2+\delta}$$

$$\times E_{\mathbf{Y}}\Big[\Big|\mathbb{1}\Big\{\Big|\widehat{U}_1 - \widehat{F}_Y^{(1)}(y)\Big| \le h_{\widehat{F}_Y^{(1)}(y)}\Big\} - \mathbb{1}\Big\{\Big|U_1 - \widehat{F}_Y^{(1)}(y)\Big| \le h_{\widehat{F}_Y^{(1)}(y)}\Big\}\Big| \times \mathbb{1}\{\mathcal{A}_n^{(1)}\}\Big]dydy',$$

by Lemma 7. Next, we take the expectation with respect to $(Y_i)_{i=1}^{n_1}$ and split the integration domain $\mathbb{R}^2$ as follows

$$\mathbb{R}^2 = \{\mathcal{D}_1 \times \mathbb{R}\} \bigcup \{\mathcal{D}_2 \times \mathbb{R}\} \bigcup \{\mathcal{D}_3 \times \mathbb{R}\} \tag{S.2.6}$$

where

$$\mathcal{D}_1 = \Big(-\infty, (\widehat{F}_Y^{(1)})^{-1}\Big(\frac{2a_n}{\sqrt{n}}\Big)\Big],$$

$$\mathcal{D}_2 = \Big((\widehat{F}_Y^{(1)})^{-1}\Big(\frac{2a_n}{\sqrt{n}}\Big), (\widehat{F}_Y^{(1)})^{-1}\Big(1 - \frac{2a_n}{\sqrt{n}}\Big)\Big),$$

$$\mathcal{D}_3 = \Big[(\widehat{F}_Y^{(1)})^{-1}\Big(1 - \frac{2a_n}{\sqrt{n}}\Big), \infty\Big).$$

Consider the second domain $\mathcal{D}_2$. By Lemma S.2.6,

$$\frac{1}{\varepsilon_n}\int_{\mathcal{D}_2} \widehat{F}_Y^{(1)}(y)^{-b_1-d_1-\delta}\widehat{F}_Y^{(2)}(y')^{d_1+\delta}\bar{\widehat{F}}_Y^{(1)}(y)^{-b_2-d_2-\delta}\bar{\widehat{F}}_Y^{(2)}(y')^{d_2+\delta}$$

$$\times \Big(\widehat{F}_Y^{(1)}(y)\Big)^{-b_1}\Big(\bar{\widehat{F}}_Y^{(1)}(y)\Big)^{-b_2}\frac{a_n}{\sqrt{n}}dydy'$$

$$= \frac{2a_n}{\varepsilon_n\sqrt{n}} \int_{(\widehat{F}_Y^{(1)})^{-1}(\frac{2a_n}{\sqrt{n}})}^{(\widehat{F}_Y^{(1)})^{-1}(1-\frac{2a_n}{\sqrt{n}})} \widehat{F}_Y^{(1)}(y)^{-2b_1-d_1-\delta} \bar{\widehat{F}}_Y^{(1)}(y)^{-2b_2-d_2-\delta} dy \int_{\mathbb{R}} \widehat{F}_Y^{(2)}(y')^{d_1+\delta} \bar{\widehat{F}}_Y^{(2)}(y')^{d_2+\delta} dy'.$$

Taking the expectation with respect to $(Y_i)_i$ and using the Cauchy–Schwarz inequality, the latter quantity is upper bounded as

$$\begin{aligned}
&\frac{2a_n}{\varepsilon_n\sqrt{n}} E^{1/2}\left[\left(\int_{(\widehat{F}_Y^{(1)})^{-1}(\frac{2a_n}{\sqrt{n}})}^{(\widehat{F}_Y^{(1)})^{-1}(1-\frac{2a_n}{\sqrt{n}})} \widehat{F}_Y^{(1)}(y)^{-2b_1-d_1-\delta} \bar{\widehat{F}}_Y^{(1)}(y)^{-2b_2-d_2-\delta} dy\right)^2\right] \\
&\quad \times E^{1/2}\left[\left(\int_{\mathbb{R}} \widehat{F}_Y^{(2)}(y')^{d_1+\delta} \bar{\widehat{F}}_Y^{(2)}(y')^{d_2+\delta} dy'\right)^2\right] \\
&\lesssim \frac{a_n}{\varepsilon_n\sqrt{n}} E^{1/2}\left[\left(\int_{(\widehat{F}_Y^{(1)})^{-1}(\frac{2a_n}{\sqrt{n}})}^{(\widehat{F}_Y^{(1)})^{-1}(1-\frac{2a_n}{\sqrt{n}})} \widehat{F}_Y^{(1)}(y)^{-2b_1-d_1-\delta} \bar{\widehat{F}}_Y^{(1)}(y)^{-2b_2-d_2-\delta} dy\right)^2\right] \text{ by Lemma 9} \\
&= o(1)
\end{aligned}$$

by Lemma S.2.7 applied with $\gamma_1 = -2b_1 - d_1 - \delta \geq \frac{1}{2} - b_1$ and $\gamma_2 = -2b_2 - d_2 - \delta \geq \frac{1}{2} - b_2$. We now consider the first domain $\mathcal{D}_1$ (the third one $\mathcal{D}_3$ can be analyzed in a similar way). We have

$$\begin{aligned}
&\frac{1}{\varepsilon_n} \int_{-\infty}^{(\widehat{F}_Y^{(1)})^{-1}(\frac{2a_n}{\sqrt{n}})} \int_{\mathbb{R}} \widehat{F}_Y^{(1)}(y)^{-b_1-d_1-\delta} \widehat{F}_Y^{(2)}(y')^{d_1+\delta} \bar{\widehat{F}}_Y^{(1)}(y)^{-b_2-d_2-\delta} \bar{\widehat{F}}_Y^{(2)}(y')^{d_2+\delta} (\widehat{F}_Y^{(1)}(y))^{1-b_1} dy dy' \\
&\lesssim \frac{1}{\varepsilon_n} \int_{-\infty}^{(\widehat{F}_Y^{(1)})^{-1}(\frac{2a_n}{\sqrt{n}})} \widehat{F}_Y^{(1)}(y)^{1-2b_1-d_1-\delta} dy \int_{\mathbb{R}} \widehat{F}_Y^{(2)}(y')^{d_1+\delta} \bar{\widehat{F}}_Y^{(2)}(y')^{d_2+\delta} dy'.
\end{aligned}$$

Moreover, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&E\left[\frac{1}{\varepsilon_n} \int_{-\infty}^{(\widehat{F}_Y^{(1)})^{-1}(\frac{2a_n}{\sqrt{n}})} \widehat{F}_Y^{(1)}(y)^{1-2b_1-d_1-\delta} dy \int_{\mathbb{R}} \widehat{F}_Y^{(2)}(y')^{d_1+\delta} \bar{\widehat{F}}_Y^{(2)}(y')^{d_2+\delta} dy'\right] \\
&\leq E^{1/2}\left[\left(\frac{1}{\varepsilon_n} \int_{-\infty}^{(\widehat{F}_Y^{(1)})^{-1}(\frac{2a_n}{\sqrt{n}})} \widehat{F}_Y^{(1)}(y)^{1-2b_1-d_1-\delta} dy\right)^2\right] E^{1/2}\left[\left(\int_{\mathbb{R}} \widehat{F}_Y^{(2)}(y')^{d_1+\delta} \bar{\widehat{F}}_Y^{(2)}(y')^{d_2+\delta} dy'\right)^2\right] \\
&= o(1)
\end{aligned}$$

by Lemmas 9 and S.2.8.

Second, we show (S.2.4). This follows from the observation that $I(y, y') = II_N(y', y)$ and the proof is analogous to the steps above.

Third, we show (S.2.5). By independence of $\widehat{U}^{(1)}$ and $\widehat{U}^{(2)}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{w(\widehat{F}_Y^{(1)}(y), \widehat{F}_Y^{(2)}(y'))}{h_{\widehat{F}_Y^{(1)}(y)} h_{\widehat{F}_Y^{(2)}(y')}} III(y, y') dy dy' \\
&= \int_{\mathbb{R}^2} \frac{w(\widehat{F}_Y^{(1)}(y), \widehat{F}_Y^{(2)}(y'))}{h_{\widehat{F}_Y^{(1)}(y)} h_{\widehat{F}_Y^{(2)}(y')}} E_{\mathbf{Y}}\Big[\Big|\mathbb{1}\Big\{\big|\widehat{U}^{(1)} - \widehat{F}_Y^{(1)}(y)\,\big| \leq h_{\widehat{F}_Y^{(1)}(y)}\Big\} - \mathbb{1}\Big\{\big|U_1 - \widehat{F}_Y^{(1)}(y)\big| \leq h_{\widehat{F}_Y^{(1)}(y)}\Big\}\Big|
\end{aligned}$$

$$\times \Big|\mathbb{1}\left\{\left|\widehat{U}^{(2)} - \widehat{F}_Y^{(2)}(y')\right| \le h_{\widehat{F}_Y^{(2)}(y')}\right\} - \mathbb{1}\left\{\left|U_2 - \widehat{F}_Y^{(2)}(y')\right| \le h_{\widehat{F}_Y^{(2)}(y')}\right\}\Big| \times \mathbb{1}\{\mathcal{A}_n\}\Big]dydy'$$

$$\le \int_{\mathbb{R}^2} \frac{(\widehat{F}_Y^{(1)}(y)\widehat{F}_Y^{(2)}(y'))^{1/2}(\bar{\widehat{F}}_Y^{(1)}(y), \bar{\widehat{F}}_Y^{(2)}(y'))^{1/2}}{h_{\widehat{F}_Y^{(1)}(y)}h_{\widehat{F}_Y^{(2)}(y')}}$$

$$\times E_{\mathbf{Y}}\Big[\Big|\mathbb{1}\left\{\left|\widehat{U}^{(1)} - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right\} - \mathbb{1}\left\{\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right\}\Big| \times \mathbb{1}\{\mathcal{A}_n^{(1)}\}\Big]$$

$$\times E_{\mathbf{Y}}\Big[\Big|\mathbb{1}\left\{\left|\widehat{U}^{(2)} - \widehat{F}_Y^{(2)}(y')\right| \le h_{\widehat{F}_Y^{(2)}(y')}\right\} - \mathbb{1}\left\{\left|U_2 - \widehat{F}_Y^{(2)}(y')\right| \le h_{\widehat{F}_Y^{(2)}(y')}\right\}\Big| \times \mathbb{1}\{\mathcal{A}_n^{(2)}\}\Big]dydy'$$

$$= \Big(\int_{\mathbb{R}} \frac{(\widehat{F}_Y^{(1)}(y)\bar{\widehat{F}}_Y^{(1)}(y))^{1/2}}{h_{\widehat{F}_Y^{(1)}(y)}} E_{\mathbf{Y}}\Big[\Big|\mathbb{1}\left\{\left|\widehat{U}^{(1)} - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right\}$$

$$- \mathbb{1}\left\{\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right\}\Big| \times \mathbb{1}\{\mathcal{A}_n^{(1)}\}\Big]dy\Big)^2$$

$$= \Big(\int_{\mathbb{R}} \frac{(\widehat{F}_Y^{(1)}(y)\bar{\widehat{F}}_Y^{(1)}(y))^{-1/2}}{\varepsilon_n} E_{\mathbf{Y}}\Big[\Big|\mathbb{1}\left\{\left|\widehat{U}^{(1)} - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right\}$$

$$- \mathbb{1}\left\{\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right\}\Big| \times \mathbb{1}\{\mathcal{A}_n^{(1)}\}\Big]dy\Big)^2$$

$$= \Big(\int_{\mathcal{D}_1\cup\mathcal{D}_2\cup\mathcal{D}_3} \frac{1}{\varepsilon_n}(\widehat{F}_Y^{(1)}(y)\bar{\widehat{F}}_Y^{(1)}(y))^{-1/2}$$

$$\times E_{\mathbf{Y}}\Big[\Big|\mathbb{1}\left\{\left|\widehat{U}^{(1)} - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right\} - \mathbb{1}\left\{\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right\}\Big| \times \mathbb{1}\{\mathcal{A}_n^{(1)}\}\Big]dy\Big)^2$$

where the domains $\mathcal{D}_i$ are defined in (S.2.6). We now show that the latter quantity tends to zero as $N \to \infty$ by successively placing ourselves on the domains $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$. Consider first the domain $\mathcal{D}_2$. By Lemma S.2.6 and the Cauchy-Schwarz inequality, we have

$$E\Big[\int_{\mathcal{D}_2} \frac{1}{\varepsilon_n}\Big(\widehat{F}_Y^{(1)}(y)\bar{\widehat{F}}_Y^{(1)}(y)\Big)^{-1/2}$$

$$\times E_{\mathbf{Y}}\Big[\Big|\mathbb{1}\left\{\left|\widehat{U}^{(1)} - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right\} - \mathbb{1}\left\{\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right\}\Big| \times \mathbb{1}\{\mathcal{A}_n^{(1)}\}\Big]dy\Big]$$

$$\lesssim E^{1/2}\left[\left(\int_{\mathcal{D}_2} \frac{1}{\varepsilon_n}(\widehat{F}_Y^{(1)}(y)\bar{\widehat{F}}_Y^{(1)}(y))^{-1/2}\frac{a_n}{\sqrt{n}}\left(\widehat{F}_Y^{(1)}(y)\right)^{-b_1}\left(\bar{\widehat{F}}_Y^{(1)}(y)\right)^{-b_2}dy\right)^2\right]$$

$$= o(1)$$

by Lemma S.2.7. We now consider the domain $\mathcal{D}_1$ (the integral over the domain $\mathcal{D}_3$ can be controlled in a similar way). We have, by Lemma S.2.6

$$E\left[\int_{\mathcal{D}_1} \frac{1}{\varepsilon_n}\Big(\widehat{F}_Y^{(1)}(y)\bar{\widehat{F}}_Y^{(1)}(y)\Big)^{-1/2} E_{\mathbf{Y}}\Big[\Big|\mathbb{1}\left\{\left|\widehat{U}^{(1)} - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right\}\right.$$

$$\left. - \mathbb{1}\left\{\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right\}\Big|\Big]dy\right]$$

$$
\begin{aligned}
&\lesssim E\left[\int_{\mathcal{D}_1} \frac{1}{\varepsilon_n}\Big(\widehat{F}_Y^{(1)}(y)\bar{\widehat{F}}_Y^{(1)}(y)\Big)^{-1/2}\widehat{F}_Y^{(1)}(y)^{1-b_1}dy\right]\\
&\lesssim E\left[\int_{\mathcal{D}_1} \frac{1}{\varepsilon_n}\widehat{F}_Y^{(1)}(y)^{\frac{1}{2}-b_1}dy\right]\\
&= o(1)
\end{aligned}
$$

by Lemma S.2.8. This concludes the proof.

Lemma S.2.6 *For any $y \in \mathbb{R}$, we have*

$$
E_{\mathbf{Y}}\Big[\Big|\mathbb{1}\left\{\left|\widehat{U}_1^{(1)} - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right\} - \mathbb{1}\left\{\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right\}\Big| \times \mathbb{1}\{\mathcal{A}_n^{(1)}\}\Big]
$$

$$
\lesssim \begin{cases}
\dfrac{a_n}{\sqrt{n}}\left(\widehat{F}_Y^{(1)}(y)\right)^{-b_1}\left(\bar{\widehat{F}}_Y^{(1)}(y)\right)^{-b_2} & \text{if } \widehat{F}_Y^{(1)}(y) \wedge \bar{\widehat{F}}_Y^{(1)}(y) \ge \frac{2a_n}{\sqrt{n}}\\
\left(\widehat{F}_Y^{(1)}(y)\right)^{1-b_1} & \text{if } \widehat{F}_Y^{(1)}(y) < \frac{2a_n}{\sqrt{n}}\\
\left(\bar{\widehat{F}}_Y^{(1)}(y)\right)^{1-b_2} & \text{if } \bar{\widehat{F}}_Y^{(1)}(y) < \frac{2a_n}{\sqrt{n}}.
\end{cases}
$$

Proof: We have

$$
\begin{aligned}
&E_{\mathbf{Y}}\Big[\Big|\mathbb{1}\left\{\left|\widehat{U}_1^{(1)} - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right\} - \mathbb{1}\left\{\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right\}\Big| \times \mathbb{1}\{\mathcal{A}_n^{(1)}\}\Big]\\
&= \Pr{}_{\mathbf{Y}}\left(\left|\widehat{U}_1^{(1)} - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)} \text{ and } \left|U_1 - \widehat{F}_Y^{(1)}(y)\right| > h_{\widehat{F}_Y^{(1)}(y)} \text{ and } \mathcal{A}_n^{(1)}\right)\\
&\quad + \Pr{}_{\mathbf{Y}}\left(\left|\widehat{U}_1^{(1)} - \widehat{F}_Y^{(1)}(y)\right| > h_{\widehat{F}_Y^{(1)}(y)} \text{ and } \left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)} \text{ and } \mathcal{A}_n^{(1)}\right).
\end{aligned}
$$

Assume first that $\widehat{F}_Y^{(1)}(y) \wedge \bar{\widehat{F}}_Y(y) \ge 2a_n/\sqrt{n}$. We have

$$
\begin{aligned}
&\Pr{}_{\mathbf{Y}}\left(\left|\widehat{U}_1^{(1)} - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)} \text{ and } \left|U_1 - \widehat{F}_Y^{(1)}(y)\right| > h_{\widehat{F}_Y^{(1)}(y)} \text{ and } \mathcal{A}_n^{(1)}\right)\\
&\le \Pr{}_{\mathbf{Y}}\left(\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)} + |\widehat{U}_1^{(1)} - U_1|\ ,\ \left|U_1 - \widehat{F}_Y^{(1)}(y)\right| > h_{\widehat{F}_Y^{(1)}(y)}\ ,\ |\widehat{U}_1^{(1)} - U_1| \le \frac{a_n}{\sqrt{n}}\right)\\
&\le \Pr{}_{\mathbf{Y}}\left(\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \in \left[h_{\widehat{F}_Y^{(1)}(y)}, h_{\widehat{F}_Y^{(1)}(y)} + \frac{a_n}{\sqrt{n}}\right]\right)\\
&\le \Pr{}_{\mathbf{Y}}\left(\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \in \left[h_{\widehat{F}_Y^{(1)}(y)} - \frac{a_n}{\sqrt{n}}, h_{\widehat{F}_Y^{(1)}(y)} + \frac{a_n}{\sqrt{n}}\right]\right).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\Pr{}_{\mathbf{Y}}\left(\left|\widehat{U}_1^{(1)} - \widehat{F}_Y^{(1)}(y)\right| > h_{\widehat{F}_Y^{(1)}(y)} \text{ and } \left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)} \text{ and } \mathcal{A}_n^{(1)}\right)\\
&\le \Pr{}_{\mathbf{Y}}\left(\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\ ,\ \left|U_1 - \widehat{F}_Y^{(1)}(y)\right| > h_{\widehat{F}_Y^{(1)}(y)} - |\widehat{U}_1^{(1)} - U_1|\ ,\ |\widehat{U}_1^{(1)} - U_1| \le \frac{a_n}{\sqrt{n}}\right)\\
&\le \Pr{}_{\mathbf{Y}}\left(\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \in \left[h_{\widehat{F}_Y^{(1)}(y)} - \frac{a_n}{\sqrt{n}}, h_{\widehat{F}_Y^{(1)}(y)}\right]\right)
\end{aligned}
$$

$$\leq \Pr_{\mathbf{Y}}\left(\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \in \left[h_{\widehat{F}_Y^{(1)}(y)} - \frac{a_n}{\sqrt{n}}, h_{\widehat{F}_Y^{(1)}(y)} + \frac{a_n}{\sqrt{n}}\right]\right).$$

Now, we have

$$\begin{aligned}
&\Pr_{\mathbf{Y}}\left(\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \in \left[h_{\widehat{F}_Y^{(1)}(y)} - \frac{a_n}{\sqrt{n}}, h_{\widehat{F}_Y^{(1)}(y)} + \frac{a_n}{\sqrt{n}}\right]\right) \\
&= F_U\left([\widehat{F}_Y^{(1)}(y) - h_{\widehat{F}_Y^{(1)}(y)} - \frac{a_n}{\sqrt{n}}, \widehat{F}_Y^{(1)}(y) - h_{\widehat{F}_Y^{(1)}(y)} + \frac{a_n}{\sqrt{n}}]\right) \\
&\quad + F_U\left([\widehat{F}_Y^{(1)}(y) + h_{\widehat{F}_Y^{(1)}(y)} - \frac{a_n}{\sqrt{n}}, \widehat{F}_Y^{(1)}(y) + h_{\widehat{F}_Y^{(1)}(y)} + \frac{a_n}{\sqrt{n}}]\right) \\
&= \left[f_U\left(\widetilde{F}_Y^{(1)}(y)\right) + f_U\left(\widetilde{F}_Y^{(2)}(y)\right)\right]\frac{2a_n}{\sqrt{n}},
\end{aligned}$$

where we used the mean-value theorem for some

$$\begin{aligned}
&\widetilde{F}_Y^{(1)}(y) \in [\widehat{F}_Y^{(1)}(y) - h_{\widehat{F}_Y^{(1)}(y)} - \frac{a_n}{\sqrt{n}}, \widehat{F}_Y^{(1)}(y) - h_{\widehat{F}_Y^{(1)}(y)} + \frac{a_n}{\sqrt{n}}] \subseteq \left[\frac{1}{6}\widehat{F}_Y(y), 1 - \frac{1}{6}\bar{\widehat{F}}_Y^{(1)}(y)\right] \\
&\widetilde{F}_Y^{(2)}(y) \in [\widehat{F}_Y^{(1)}(y) + h_{\widehat{F}_Y^{(1)}(y)}, \widehat{F}_Y^{(1)}(y) + h_{\widehat{F}_Y^{(1)}(y)} + \frac{a_n}{\sqrt{n}}] \subseteq \left[\widehat{F}_Y(y), 1 - \frac{1}{6}\bar{\widehat{F}}_Y^{(1)}(y)\right],
\end{aligned}$$

for N sufficiently large. By Assumption 2(iii), the latter quantity is further upper-bounded as

$$\begin{aligned}
&2\left[f_U\left(\widetilde{F}_Y^{(1)}(y)\right) + f_U\left(\widetilde{F}_Y^{(2)}(y)\right)\right]\frac{a_n}{\sqrt{n}} \\
&\leq 2C_U\left\{\left(\widetilde{F}_Y^{(1)}(y)\right)^{-b_1}\left(1 - \widetilde{F}_Y^{(1)}(y)\right)^{-b_2} + \left(\widetilde{F}_Y^{(2)}(y)\right)^{-b_1}\left(1 - \widetilde{F}_Y^{(2)}(y)\right)^{-b_2}\right\}\frac{a_n}{\sqrt{n}} \\
&\leq 4C_U\left(\frac{1}{6}\widehat{F}_Y^{(1)}(y)\right)^{-b_1}\left(\frac{1}{6}\bar{\widehat{F}}_Y^{(1)}(y)\right)^{-b_2}\frac{a_n}{\sqrt{n}} \\
&\leq C\left(\widehat{F}_Y^{(1)}(y)\right)^{-b_1}\left(\bar{\widehat{F}}_Y^{(1)}(y)\right)^{-b_2}\frac{a_n}{\sqrt{n}}.
\end{aligned}$$

This proves the desired result in the case where $\widehat{F}_Y^{(1)}(y) \wedge \bar{\widehat{F}}_Y^{(1)}(y) \geq \frac{2a_n}{\sqrt{n}}$.

Assume now that $\widehat{F}_Y^{(1)}(y) \vee \bar{\widehat{F}}_Y^{(1)}(y) < 2\frac{a_n}{\sqrt{n}}$. By symmetry, assume that $\widehat{F}_Y^{(1)}(y) < 2a_n/\sqrt{n}$. We have

$$\begin{aligned}
&\Pr_{\mathbf{Y}}\left(\left|\widehat{U}_1^{(1)} - \widehat{F}_Y^{(1)}(y)\right| > h_{\widehat{F}_Y^{(1)}(y)} \text{ and } \left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \leq h_{\widehat{F}_Y^{(1)}(y)} \text{ and } \mathcal{A}_n^{(1)}\right) \\
&\leq \Pr_{\mathbf{Y}}\left(\left|U_1 - \widehat{F}_Y^{(1)}(y)\right| \leq h_{\widehat{F}_Y^{(1)}(y)}\right) \\
&\leq \Pr_{\mathbf{Y}}\left(U_1 \leq 2\widehat{F}_Y^{(1)}(y)\right) \\
&\leq \int_0^{2\widehat{F}_Y^{(1)}(y)} C_U t^{-b_1}(1-t)^{-b_2} dt \\
&\lesssim \left(\widehat{F}_Y^{(1)}(y)\right)^{1-b_1}. \qquad \text{(S.2.7)}
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\Pr_{\mathbf{Y}}\left(\left|\widehat{U}_1^{(1)} - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)} \text{ and } \left|U_1 - \widehat{F}_Y^{(1)}(y)\right| > h_{\widehat{F}_Y^{(1)}(y)} \text{ and } \mathcal{A}_n^{(1)}\right)\\
&\le \Pr_{\mathbf{Y}}\left(\left|\widehat{U}_1^{(1)} - \widehat{F}_Y^{(1)}(y)\right| \le h_{\widehat{F}_Y^{(1)}(y)}\right)\\
&\le \Pr_{\mathbf{Y}}\left(\widehat{U}_1^{(1)} \le 2\widehat{F}_Y^{(1)}(y)\right). \qquad \text{(S.2.8)}
\end{aligned}$$

Defining $n_3' = n_3/2$, we have, for all $t \in [0,1]$,

$$\begin{aligned}
\Pr_{\mathbf{Y}}(\widehat{U}_1^{(1)} \le t) &= E_{\mathbf{Y}}\left[\Pr_{\mathbf{Y}}\left(\widehat{F}_Z^{(1)}(X_1) \le t \,|\, X_1\right)\right]\\
&= E_{\mathbf{Y}}\Big[\Pr_{\mathbf{Y}}(\text{Binomial}(n_3', F_Z(X_1)) \le n_3' t \,|\, X_1)\Big].
\end{aligned}$$

We place ourselves conditionally on X_1. Suppose first that $t > F_Z(X_1)$. Then we have

$$\Pr_{\mathbf{Y}}(\text{Binomial}(n_3', F_Z(X_1)) \le n_3' t \,|\, X_1) \le 1.$$

Otherwise, by Bennett's inequality (e.g., Theorem 2.9 in Boucheron et al., 2013), for all $t \le F_Z(X_1)$,

$$\begin{aligned}
&\Pr_{\mathbf{Y}}(\text{Binomial}(n_3', F_Z(X_1)) \le n_3' t \,|\, X_1) \le \exp\left(-n_3' F_Z(X_1)\, h\Big(\frac{n_3'(t - F_Z(X_1))}{n_3' F_Z(X_1)}\Big)\right)\\
&\text{i.e. } \Pr_{\mathbf{Y}}(\text{Binomial}(n_3', U_1) \le n_3' t \,|\, X_1) = \exp\left(-n_3' U_1\, h\Big(\frac{t - U_1}{U_1}\Big)\right),
\end{aligned}$$

where $h(x) = (1+x)\log(1+x) - x$. Taking expectation on both sides, Assumption 2(iii) yields for $t < 1/4$

$$\begin{aligned}
\Pr_{\mathbf{Y}}(\widehat{U}_1 \le t) &\le \int_0^{2t} f_U(u)du + \int_{2t}^1 \exp\left(-n_3' u\, h\Big(\frac{t-u}{u}\Big)\right) f_U(u)du\\
&\lesssim \int_0^{2t} u^{-b_1}du + \int_{2t}^{1/2} \exp\left(-n_3' u\, h\Big(\frac{t-u}{u}\Big)\right) u^{-b_1}du\\
&\qquad + \int_{1/2}^1 \exp\left(-n_3' u\, h\Big(\frac{t-u}{u}\Big)\right) (1-u)^{-b_2}du\\
&\lesssim t^{1-b_1} + \int_{2t}^{1/2} \exp\left(-cn_3' u\right) u^{-b_1}du\\
&\qquad + \int_{1/2}^1 \exp\left(-cn_3' u\right) (1-u)^{-b_2}du\\
&\lesssim t^{1-b_1} + n_3'^{b_1-1}\int_{2tcn_3'}^{\infty} \exp(-z)z^{-b_1}dz + n_3'^{b_2-1}\exp(-cn_3')\int_{cn_3'/2}^{\infty} \exp(-z)z^{-b_2}dz\\
&\lesssim t^{1-b_1} + {n_3'}^{b_1-1}\exp(-2tcn_3') + {n_3'}^{b_2-1}\exp\left(-\frac{3}{2}cn_3'\right).
\end{aligned}$$

For all $t \ge 1/n_3'$,

$${n_3'}^{b_1-1}\exp(-2tcn_3') \le t^{1-b_1}.$$

Moreover, for N larger than a constant depending on b_1 and b_2,

$$ {n_3'}^{b_2-1} \exp\left(-\frac{3}{2} c n_3'\right) \le t^{1-b_1}. $$

It follows that, for all $t \ge 1/n_3'$,

$$ \Pr_{\mathbf{Y}}(\widehat{U}_1 \le t) \lesssim t^{1-b_1}. $$

By taking $t = \widehat{F}_Y^{(1)}(y)$ and combining (S.2.7) and (S.2.8), the result follows. $\square$

Lemma S.2.7 *Let* $\gamma_1 \ge -\frac{1}{2} - b_1$ *and* $\gamma_2 \ge -\frac{1}{2} - b_2$*. Then it holds that*

$$ \frac{a_n}{\varepsilon_n \sqrt{n}} E^{1/2}\left[\left(\int_{(\widehat{F}_Y^{(1)})^{-1}(\frac{2a_n}{\sqrt{n}})}^{(\widehat{F}_Y^{(1)})^{-1}(1-\frac{2a_n}{\sqrt{n}})} \widehat{F}_Y^{(1)}(y)^{\gamma_1} \bar{\widehat{F}}_Y^{(1)}(y)^{\gamma_2} dy\right)^2\right] = o(1). $$

Proof:
We have

$$ \begin{aligned} & \frac{a_n}{\varepsilon_n \sqrt{n}} E^{1/2}\left[\left(\int_{(\widehat{F}_Y^{(1)})^{-1}(\frac{2a_n}{\sqrt{n}})}^{(\widehat{F}_Y^{(1)})^{-1}(1-\frac{2a_n}{\sqrt{n}})} \widehat{F}_Y^{(1)}(y)^{\gamma_1} \bar{\widehat{F}}_Y^{(1)}(y)^{\gamma_2} dy\right)^2\right] \\ \lesssim\ & \frac{a_n}{\varepsilon_n \sqrt{n}} \Big\{ E^{1/2}\left[\left(\int_{(\widehat{F}_Y^{(1)})^{-1}(\frac{2a_n}{\sqrt{n}})}^{(\widehat{F}_Y^{(1)})^{-1}(1/2)} \widehat{F}_Y^{(1)}(y)^{\gamma_1} \bar{\widehat{F}}_Y^{(1)}(y)^{\gamma_2} dy\right)^2\right] \\ & \quad + E^{1/2}\left[\left(\int_{(\widehat{F}_Y^{(1)})^{-1}(1/2)}^{(\widehat{F}_Y^{(1)})^{-1}(1-\frac{2a_n}{\sqrt{n}})} \widehat{F}_Y^{(1)}(y)^{\gamma_1} \bar{\widehat{F}}_Y^{(1)}(y)^{\gamma_2} dy\right)^2\right] \Big\} \\ \lesssim\ & \frac{a_n}{\varepsilon_n \sqrt{n}} \Big\{ E^{1/2}\left[\left(\int_{(\widehat{F}_Y^{(1)})^{-1}(\frac{2a_n}{\sqrt{n}})}^{(\widehat{F}_Y^{(1)})^{-1}(1/2)} \widehat{F}_Y^{(1)}(y)^{\gamma_1} dy\right)^2\right] + E^{1/2}\left[\left(\int_{(\widehat{F}_Y^{(1)})^{-1}(1/2)}^{(\widehat{F}_Y^{(1)})^{-1}(1-\frac{2a_n}{\sqrt{n}})} \bar{\widehat{F}}_Y^{(1)}(y)^{\gamma_2} dy\right)^2\right] \Big\} \\ \le\ & \frac{a_n}{\varepsilon_n \sqrt{n}} \Big\{ E^{1/2}\left[\left(\Big(\frac{2a_n}{\sqrt{n}}\Big)^{\gamma_1} \times \Big((\widehat{F}_Y^{(1)})^{-1}\big(\frac{1}{2}\big) - (\widehat{F}_Y^{(1)})^{-1}\big(\frac{2a_n}{\sqrt{n}}\big)\Big)\right)^2\right] \\ & \quad + E^{1/2}\left[\left(\Big(\frac{2a_n}{\sqrt{n}}\Big)^{\gamma_2} \times \Big((\widehat{F}_Y^{(1)})^{-1}\big(1 - \frac{2a_n}{\sqrt{n}}\big)\Big) - (\widehat{F}_Y^{(1)})^{-1}\big(\frac{1}{2}\big)\Big)\right)^2\right] \Big\}. \end{aligned} $$

We handle the first expectation (the second one can be analyzed in a similar way). We have

$$ \frac{1}{\varepsilon_n} \left(\frac{a_n}{\sqrt{n}}\right)^{1+\gamma_1} E^{1/2}\left[\left((\widehat{F}_Y^{(1)})^{-1}\big(\frac{1}{2}\big) - (\widehat{F}_Y^{(1)})^{-1}\big(\frac{2a_n}{\sqrt{n}}\big)\right)^2\right]. \tag{S.2.9} $$

Letting $n_1' = n_1/2$, we recall that the density of $\xi_{(r)}$ is (see, e.g., Equation (2.1.6) in David and Nagaraja, 2004):

$$ f_{\xi_{(r)}}(x) = \frac{1}{\mathrm{B}(r, n_1' - r + 1)} x^r \bar{x}^{n_1' - r}. $$

Combining this with Assumption 2(ii) and letting $r = \lceil 2a_n\sqrt{n}\rceil$, we have

$$
\begin{aligned}
E\left[\Big((\widehat{F}_Y^{(1)})^{-1}\Big(\frac{2a_n}{\sqrt{n}}\Big)\Big)^2\right] &= E\left[(F_Y^{-1}(\xi_{(r)}))^2\right] \\
&\lesssim \int_0^1 x^{-2d_1}\bar{x}^{-2d_2}\frac{1}{\mathrm{B}(r, n_1 - r + 1)}x^{r-1}\bar{x}^{n_1'-r}dx \\
&= \frac{\mathrm{B}(\lceil 2a_n\sqrt{n}\rceil - 2d_1, n_1' - \lceil 2a_n\sqrt{n}\rceil - 2d_2 + 1)}{\mathrm{B}(\lceil 2a_n\sqrt{n}\rceil, n_1' - \lceil 2a_n\sqrt{n}\rceil + 1)} \\
&= \frac{\Gamma(\lceil 2a_n\sqrt{n}\rceil - 2d_1)\Gamma(n_1' - \lceil 2a_n\sqrt{n}\rceil - 2d_2 + 1)}{\Gamma(n_1' + 1 - 2(d_1 + d_2))} \\
&\quad \times \frac{\Gamma(n_1' + 1)}{\Gamma(\lceil 2a_n\sqrt{n}\rceil)\Gamma(n_1' - \lceil 2a_n\sqrt{n}\rceil + 1)} \\
&\lesssim (2a_n\sqrt{n})^{-2d_1}{n_1'}^{-2d_2}{n_1'}^{2(d_1+d_2)} \qquad \text{by Lemma S.3.6} \\
&\lesssim \left(\frac{a_n}{\sqrt{n}}\right)^{-2d_1}.
\end{aligned}
$$

By a similar reasoning, there exists a constant $C > 0$ independent of N such that

$$
E\left[\Big((\widehat{F}_Y^{(1)})^{-1}\Big(\frac{1}{2}\Big)\Big)^2\right] \le C.
$$

Plugging this result into (S.2.9) yields

$$
\begin{aligned}
&\frac{1}{\varepsilon_n}\left(\frac{a_n}{\sqrt{n}}\right)^{1+\gamma_1} E^{1/2}\left[\Big((\widehat{F}_Y^{(1)})^{-1}\Big(\frac{1}{2}\Big) - (\widehat{F}_Y^{(1)})^{-1}\Big(\frac{2a_n}{\sqrt{n}}\Big)\Big)^2\right] \\
&\lesssim \frac{1}{\varepsilon_n}\left(\frac{a_n}{\sqrt{n}}\right)^{1+\gamma_1}\left(\frac{a_n}{\sqrt{n}}\right)^{-d_1} \\
&= \frac{1}{\varepsilon_n}\left(\frac{a_n}{\sqrt{n}}\right)^{1+\gamma_1-d_1} \\
&= o(1)
\end{aligned}
$$

since $1 + \gamma_1 - d_1 \ge \frac{1}{2} - b_1 - d_1 > 0$ and $a_n = O(\log(n))$. Similarly,

$$
\frac{1}{\varepsilon_n}\left(\frac{a_n}{\sqrt{n}}\right)^{1+\gamma_2} E^{1/2}\left[\Big((\widehat{F}_Y^{(1)})^{-1}\Big(1 - \frac{2a_n}{\sqrt{n}}\Big) - (\widehat{F}_Y^{(1)})^{-1}\Big(\frac{1}{2}\Big)\Big)^2\right] = o(1).
$$

$\square$

Lemma S.2.8 *For any $\alpha \ge \frac{1}{2} - b_1$, it holds that*

$$
E\left[\left(\int_{-\infty}^{(\widehat{F}_Y^{(1)})^{-1}(\frac{2a_n}{\sqrt{n}})} \widehat{F}_Y^{(1)}(y)^{\alpha}dy\right)^2\right] = o(\varepsilon_n^2).
$$

Proof: Let $n_1' = n_1/2$. We have

$$
\begin{aligned}
& E\left[\left(\int_{-\infty}^{(\widehat{F}_Y^{(1)})^{-1}(\frac{2a_n}{\sqrt{n}})} \widehat{F}_Y^{(1)}(y)^\alpha dy\right)^2\right] \\
&= E\left[\left(\sum_{i=1}^{\lceil 2\sqrt{n}a_n\rceil - 1} \left(\frac{i}{n_1'}\right)^\alpha \left(Y^{(1)}_{(i+1)} - Y^{(1)}_{(i)}\right)\right)^2\right] \\
&= \sum_{i,j=1}^{\lceil 2\sqrt{n}a_n\rceil - 1} \left(\frac{ij}{(n_1')^2}\right)^\alpha E\left[\left(Y^{(1)}_{(i+1)} - Y^{(1)}_{(i)}\right)\left(Y^{(1)}_{(j+1)} - Y^{(1)}_{(j)}\right)\right] \\
&\leq \sum_{i,j=1}^{\lceil 2\sqrt{n}a_n\rceil - 1} \left(\frac{ij}{(n_1')^2}\right)^\alpha E^{1/2}\left[\left(Y^{(1)}_{(i+1)} - Y^{(1)}_{(i)}\right)^2\right] E^{1/2}\left[\left(Y^{(1)}_{(j+1)} - Y^{(1)}_{(j)}\right)^2\right] \\
&= \left(\sum_{i=1}^{\lceil 2\sqrt{n}a_n\rceil - 1} \left(\frac{i}{n_1'}\right)^\alpha E^{1/2}\left[\left(Y^{(1)}_{(i+1)} - Y^{(1)}_{(i)}\right)^2\right]\right)^2 \\
&\lesssim \left(\frac{1}{n_1'} \sum_{i=1}^{\lceil 2\sqrt{n}a_n\rceil - 1} \left(\frac{i}{n_1'}\right)^{\alpha - (1+d_1)}\right)^2 \\
&= \left(\frac{1}{(n_1')^{\alpha - d_1}} \sum_{i=1}^{\lceil 2\sqrt{n}a_n\rceil - 1} i^{(\alpha - d_1) - 1}\right)^2 \\
&\asymp \left(\frac{a_n\sqrt{n}}{n}\right)^{2(\alpha - d_1)} \\
&= o(\varepsilon_n^2),
\end{aligned}
$$

where the first inequality follows from the Cauchy–Schwarz inequality, the asymptotic inequality follows from Lemma S.3.5, and the asymptotic equivalence from $\alpha - d_1 > 0$.

□

S.3 Other technical lemmas

Lemma S.3.1 *Let $n_1' = n_1/2$. For all $k \in \{1, \ldots, n_1' - 1\}$, we have*

$$
E\left[\left(Y^{(1)}_{(\lceil k+7n_1' h_{k/n_1'}\rceil)} - Y^{(1)}_{(\lceil k-7n_1' h_{k/n_1'}\rceil)}\right)^2\right] \lesssim \left(\frac{h_{k/n_1'} + \frac{1}{n_1'}}{\left(\frac{k}{n_1'}\right)^{1+d_1}\left(\frac{n_1'-k}{n_1'}\right)^{1+d_2}}\right)^2.
$$

Proof: Let $a_k = \lceil k - 7n_1' h_{k/n_1'}\rceil$ and $b_k = \lceil k + 7n_1' h_{k/n_1'}\rceil$. We note that we always have $b_k \geq a_k + 1$ for any $k \in \{1, \ldots, n_1' - 1\}$. Suppose that $a_k \geq 10$ and $b_k \leq n_1' - 10$. Then, we have $a_k \asymp b_k \asymp k$ and $b_k - a_k \lesssim n_1' h_{k/n_1'}$. The result follows from Lemma S.3.5.1.

Now, suppose that $a_k \leq 10$ or $n_1' - b_k \leq 10$. By symmetry, suppose $a_k \leq 10$. By the definition of $h_{k/n_1'}$, we have $7n_1' h_{k/n_1'} < k/12$ for n_1' large enough, so that

$$\lceil k - 7n_1' h_{k/n_1'} \rceil \leq 10 \implies \lceil 11k/12 \rceil \leq 10 \implies k \leq 10.$$

In particular, we have $7n_1' h_{k/n_1'} < k/12 < 1$, which implies $b_k = a_k + 1$. The result follows from Lemma S.3.5.2. $\square$

Lemma S.3.2 *For any $y \in \mathbb{R}$ and $0 \leq b_1, b_2 \leq 1/2$, we have for all $j \in \{1,2\}$:*

$$E^{1/2}\left[\left|\widehat{F}_Y^{(j)}(y) - F_Y(y)\right|^{2(1-2b_1)} \mathbb{1}_{\left\{\widehat{F}_Y^{(j)}(y)\bar{\widehat{F}}_Y^{(j)}(y) \neq 0\right\}}\right] \leq \frac{2^{1/2-b_1}\left(F_Y(y)\bar{F}_Y(y)\right)^{1/2}}{n_1^{1/2-2b_1}} \wedge \left(\frac{2F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{1/2-b_1}$$
$$E^{1/2}\left[\left|\bar{\widehat{F}}_Y^{(j)}(y) - \bar{F}_Y(y)\right|^{2(1-2b_2)} \mathbb{1}_{\left\{\widehat{F}_Y^{(j)}(y)\bar{\widehat{F}}_Y^{(j)}(y) \neq 0\right\}}\right] \leq \frac{2^{1/2-b_2}\left(F_Y(y)\bar{F}_Y(y)\right)^{1/2}}{n_1^{1/2-2b_2}} \wedge \left(\frac{2F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{1/2-b_2}.$$

Proof: To ease notation, we drop the superscript (j) and simply write $\widehat{F}_Y$. We define the weights $u = 1/(1-2b_1)$ and $v = 1/(2b_1)$ that satisfy $\frac{1}{u} + \frac{1}{v} = 1$ and $u, v \geq 1$, using the convention that $1/0 = \infty$ if $b_1 = 1/2$. By the Hölder inequality, we have

$$\begin{aligned}
& E^{1/2}\left[\left|\widehat{F}_Y(y) - F_Y(y)\right|^{2(1-2b_1)} \mathbb{1}_{\left\{\widehat{F}_Y(y)\bar{\widehat{F}}_Y(y) \neq 0\right\}}\right] \\
&\leq E^{1/(2u)}\left[\left|\widehat{F}_Y(y) - F_Y(y)\right|^2\right] E^{1/(2v)}\left[\left(\mathbb{1}_{\left\{\widehat{F}_Y(y)\bar{\widehat{F}}_Y(y) \neq 0\right\}}\right)^v\right] \\
&\leq \left(\frac{2F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{\frac{1-2b_1}{2}} \left(\Pr\left(\widehat{F}_Y(y) \neq 0\right) \wedge \Pr\left(\bar{\widehat{F}}_Y(y) \neq 0\right)\right)^{b_1} \\
&\leq \left(\frac{2F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{\frac{1-2b_1}{2}} \left\{\left(1 \wedge \left(\frac{n_1 F_Y(y)}{2}\right)\right)^{b_1} \wedge \left(1 \wedge \left(\frac{n_1 \bar{F}_Y(y)}{2}\right)\right)^{b_1}\right\} \\
&= \left(\frac{2F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{\frac{1-2b_1}{2}} n_1^{b_1}\left\{\frac{F_Y(y) \wedge \bar{F}_Y(y)}{2} \wedge \frac{1}{n_1}\right\}^{b_1} \\
&\leq \left(\frac{2F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{\frac{1-2b_1}{2}} n_1^{b_1}\left\{F_Y(y)\bar{F}_Y(y) \wedge \frac{1}{n_1}\right\}^{b_1} \qquad \text{using } u \wedge \bar{u} \leq 2u\bar{u},\ \forall u \in [0,1] \\
&= \frac{2^{1/2-b_1}\left(F_Y(y)\bar{F}_Y(y)\right)^{1/2}}{n_1^{1/2-2b_1}} \wedge \left(\frac{2F_Y(y)\bar{F}_Y(y)}{n_1}\right)^{1/2-b_1},
\end{aligned}$$

which proves the first claim. The second claim can be proved in a similar fashion, following the same steps as outlined above. $\square$

Lemma S.3.3 *For any $x, y \in \mathbb{R}$ and $\alpha \in (0,1]$, it holds that*

$$\begin{aligned}
\left|\left(\widehat{F}_Y^{(1)}(x) \wedge \widehat{F}_Y^{(2)}(y)\right)^\alpha - \left(F_Y(x) \wedge F_Y(y)\right)^\alpha\right| \leq\ & \left|\widehat{F}_Y^{(1)}(x) - F_Y(x)\right|^\alpha \wedge F_Y(y)^\alpha \\
& + \left|\widehat{F}_Y^{(2)}(y) - F_Y(y)\right|^\alpha \wedge F_Y(x)^\alpha
\end{aligned}$$

$$+\left|\widehat{F}_Y^{(1)}(x) - F_Y(x)\right|^\alpha \wedge \left|\widehat{F}_Y^{(2)}(y) - F_Y(y)\right|^\alpha,$$

and the same inequality holds true by replacing $\widehat{F}_Y^{(1)}$, $\widehat{F}_Y^{(2)}$, *and* F_Y *with* $\bar{\widehat{F}}_Y^{(1)}$, $\bar{\widehat{F}}_Y^{(2)}$, *and* $\bar{F}_Y$, *respectively.*

Proof: Below, we will use the classical inequalities that hold true for any $a, b, c \geq 0$ and $\alpha \in (0,1]$

$$\begin{aligned}&|a \wedge b - a \wedge c| \leq a \wedge |b - c|\\&|a^\alpha - b^\alpha| \leq |a - b|^\alpha.\end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned}\left|(\widehat{a} \wedge \widehat{b})^\alpha - (a \wedge b)^\alpha\right| &\leq \Big(\left|(\widehat{a} \wedge \widehat{b})^\alpha - (a \wedge \widehat{b})^\alpha\right|\Big) + \Big(\left|(a \wedge \widehat{b})^\alpha - (a \wedge b)^\alpha\right|\Big)\\&\leq \Big(\left|\widehat{a}^\alpha - a^\alpha\right| \wedge \widehat{b}^\alpha\Big) + \Big(a^\alpha \wedge \left|\widehat{b}^\alpha - b^\alpha\right|\Big)\\&\leq \Big(\left|\widehat{a} - a\right|^\alpha \wedge \widehat{b}^\alpha\Big) + \Big(a^\alpha \wedge \left|\widehat{b} - b\right|^\alpha\Big)\\&\leq \Big(\left|\widehat{a} - a\right|^\alpha \wedge b^\alpha\Big) + \Big(a^\alpha \wedge \left|\widehat{b} - b\right|^\alpha\Big)\\&\quad + \left|\widehat{a} - a\right|^\alpha \wedge \left|\widehat{b} - b\right|^\alpha.\end{aligned}$$

Applying this with $a = F_Y(x)$, $\hat{a} = \widehat{F}_Y^{(1)}(x)$, $b = F_Y(y)$, and $\hat{b} = \widehat{F}_Y^{(2)}(y)$ yields the result. ☐

Lemma S.3.4 *For any* $x, y \in \mathbb{R}$ *and* $\alpha \in (0,1]$, *it holds that*

$$E\left[\left|(\widehat{F}_Y^{(1)}(x) \wedge \widehat{F}_Y^{(2)}(y))^\alpha - (F_Y(x) \wedge F_Y(y))^\alpha\right|\right] \leq 6(F_Y(x) \wedge F_Y(y))^\alpha$$

and $E^{1/2}\left[\left|(\widehat{F}_Y^{(1)}(x) \wedge \widehat{F}_Y^{(2)}(y))^\alpha - (F_Y(x) \wedge F_Y(y))^\alpha\right|\right] \to 0$ *as* $N \to \infty$.

Proof: By Lemma S.3.3, we have

$$\begin{aligned}E\left[\left|(\widehat{F}_Y^{(1)}(x) \wedge \widehat{F}_Y^{(2)}(y))^\alpha - (F_Y(x) \wedge F_Y(y))^\alpha\right|\right] \leq \; & E\left[\left|\widehat{F}_Y^{(1)}(x) - F_Y(x)\right|^\alpha\right] \wedge E\left[\left|\widehat{F}_Y^{(2)}(y) - F_Y(y)\right|^\alpha\right]\\&+ E\left[\left|\widehat{F}_Y^{(1)}(x) - F_Y(x)\right|^\alpha\right] \wedge F_Y(y)^\alpha\\&+ E\left[\left|\widehat{F}_Y^{(2)}(y) - F_Y(y)\right|^\alpha\right] \wedge F_Y(x)^\alpha. \qquad \text{(S.3.1)}\end{aligned}$$

We now control the quantity $E\left[|\widehat{F}_Y^{(1)}(x) - F_Y(x)|^\alpha\right]$ for any fixed $x \in \mathbb{R}$. We define the weights $v = 2/\alpha \geq 1$ and $u = \frac{v}{v-1}$. These quantities satisfy $u \leq 2/\alpha$ since $v - 1 \geq 1$, and $\frac{1}{u} + \frac{1}{v} = 1$. By

Hölder's inequality, we therefore have

$$
\begin{aligned}
E\left[\left|\widehat{F}_Y^{(1)}(x) - F_Y(x)\right|^{\alpha}\right] &= E\left[\left|\widehat{F}_Y^{(1)}(x) - F_Y(x)\right|^{\alpha}\mathbb{1}_{\widehat{F}_Y^{(1)}(x)\neq 0} + F_Y(x)^{\alpha}\,\mathbb{1}_{\widehat{F}_Y^{(1)}(x)=0}\right] \\
&\leq E^{1/u}\left[\left|\widehat{F}_Y^{(1)}(x) - F_Y(x)\right|^{\alpha u}\right]\operatorname{Pr}\left(\mathbb{1}_{\widehat{F}_Y^{(1)}(x)\neq 0}\right)^{1/v} + F_Y(x)^{\alpha}\operatorname{Pr}\left(\mathbb{1}_{\widehat{F}_Y^{(1)}(x)=0}\right) \\
&\leq E^{\alpha/2}\left[\left|\widehat{F}_Y^{(1)}(x) - F_Y(x)\right|^{2}\right]\left(1 - \bar{F}_Y(x)^{n_1/2}\right)^{1/v} + F_Y(x)^{\alpha}\bar{F}_Y(x)^{n_1/2} \\
&\leq \left(\frac{2F_Y(x)\bar{F}_Y(x)}{n_1}\right)^{\alpha/2}\left(1 \wedge \left(\frac{n_1}{2}F_Y(x)\right)\right)^{1/v} + F_Y(x)^{\alpha}\bar{F}_Y(x)^{n_1/2} \\
&\leq \left(\frac{2F_Y(x)}{n_1}\right)^{\alpha/2} \wedge F_Y(x)^{\alpha} + F_Y(x)^{\alpha}\bar{F}_Y(x)^{n_1/2},
\end{aligned}
$$

where we used Jensen's inequality in the third line (which can be applied here since $\alpha u/2 \leq 1$) and the inequality $1-(1-a)^n \leq 1 \wedge (na)$ for any $a \in [0,1]$ and $n \in \mathbb{N}$ in the fourth line. It follows from the display above that $E\left[\left|\widehat{F}_Y^{(1)}(x) - F_Y(x)\right|^{\alpha}\right] \leq 2F_Y(x)^{\alpha}$ and that $E\left[\left|\widehat{F}_Y^{(1)}(x) - F_Y(x)\right|^{\alpha}\right] \to 0$ as $n_1 \to \infty$ for any fixed x. By swapping the roles of x and y, we similarly obtain that $E\left[\left|\widehat{F}_Y^{(2)}(y) - F_Y(y)\right|^{\alpha}\right] \leq 2F_Y(y)^{\alpha}$ and that $E\left[\left|\widehat{F}_Y^{(2)}(y) - F_Y(y)\right|^{\alpha}\right] \to 0$ as $n_1 \to \infty$ for any fixed y. Combining with (S.3.1) yields that

$$
E\left[\left|\left(\widehat{F}_Y^{(1)}(x) \wedge \widehat{F}_Y^{(2)}(y)\right)^{\alpha} - \left(F_Y(x) \wedge F_Y(y)\right)^{\alpha}\right|\right] \leq 6\left(F_Y(x) \wedge F_Y(y)\right)^{\alpha}
$$

and that $E\left[\left|\left(\widehat{F}_Y^{(1)}(x) \wedge \widehat{F}_Y^{(2)}(y)\right)^{\alpha} - \left(F_Y(x) \wedge F_Y(y)\right)^{\alpha}\right|\right] \to 0$ as $N \to \infty$ for any fixed $x, y \in \mathbb{R}$, which concludes the proof. □

Lemma S.3.5

1. *For all $a, b \in \{10, \ldots, n_1/2 - 10\}$ such that $a < b$,*

$$
E\left[\left(Y_{(b)}^{(1)} - Y_{(a)}^{(1)}\right)^2\right] \lesssim \left[\left(\frac{a}{n_1}\right)^{-(1+d_1)}\left(\frac{n_1 - b}{n_1}\right)^{-(1+d_2)}\left(\frac{b-a}{n_1}\right)\right]^2.
$$

2. *For all $k \in \{1, \ldots, 10\} \cup \{n_1/2 - 11, \ldots, n_1/2 - 1\}$:*

$$
E\left[\left(Y_{(k+1)}^{(1)} - Y_{(k)}^{(1)}\right)^2\right] \lesssim n_1^{2(\mathbb{1}\{k \leq 10\}d_1 + \mathbb{1}\{k \geq n_1/2 - 11\}d_2)}.
$$

Proof: For ease of notation we drop the superscript (1) and write $Y_{(a)}, Y_{(b)}$ instead of $Y_{(a)}^{(1)}, Y_{(b)}^{(1)}$.

1. Suppose that $a \geq 10$ and $b \leq n_1/2 - 10$ such that $a < b$.

$$
\begin{aligned}
E\left[\left(Y_{(b)} - Y_{(a)}\right)^2\right] &= E\left[\left(F_Y^{-1}\left(\xi_{(b)}\right) - F_Y^{-1}\left(\xi_{(a)}\right)\right)^2\right] \\
&= E\left[\left((F_Y^{-1})'(\tilde{\xi})\right)^2\left(\xi_{(b)} - \xi_{(a)}\right)^2\right]
\end{aligned}
$$

$$\leq E\left[\tilde{\xi}^{-2(1+d_1)}(1-\tilde{\xi})^{-2(1+d_2)}\left(\xi_{(b)}-\xi_{(a)}\right)^2\left(\mathbb{1}\left\{\tilde{\xi}\geq 1/2\right\}+\mathbb{1}\left\{\tilde{\xi}\leq 1/2\right\}\right)\right],$$

for some $\tilde{\xi}\in(\xi_{(a)},\xi_{(b)})$. We bound $E\left[\tilde{\xi}^{-2(1+d_1)}(1-\tilde{\xi})^{-2(1+d_2)}\left(\xi_{(b)}-\xi_{(a)}\right)^2\mathbb{1}\left\{\tilde{\xi}\leq 1/2\right\}\right]$, and the second term in the sum can be analyzed in a similar way. We have

$$\begin{aligned}
&E\left[\tilde{\xi}^{-2(1+d_1)}(1-\tilde{\xi})^{-2(1+d_2)}\left(\xi_{(b)}-\xi_{(a)}\right)^2\mathbb{1}\left\{\tilde{\xi}\leq 1/2\right\}\right]\\
&\leq 2^{2(1+d_2)}E\left[\left(\xi_{(a)}\right)^{-2(1+d_1)}\left(\xi_{(b)}-\xi_{(a)}\right)^2\right]\\
&\leq 2^{2d}E^{1/2}\left[\left(\xi_{(a)}\right)^{-4(1+d_1)}\right]E^{1/2}\left[\left(\xi_{(b)}-\xi_{(a)}\right)^4\right]. \qquad \text{(S.3.2)}
\end{aligned}$$

We compute the two expectations in the right-hand side separately. We recall that the density of $\xi_{(a)}$ at point $u\in(0,1)$ is given by

$$f_{\xi_{(a)}}(u)=\frac{n_1!}{(a-1)!(n_1-a)!}u^{a-1}(1-u)^{n_1-a}.$$

Therefore, since $a\geq 10>4(1+d_1)$, we obtain

$$\begin{aligned}
E\left[\xi_{(a)}^{-4(1+d_1)}\right]&=\int_{(0,1)}\frac{n_1!}{(a-1)!(n_1-a)!}u^{a-1-4(1+d_1)}(1-u)^{n_1-a}du\\
&=\frac{n_1!}{(a-1)!(n_1-a)!}\mathrm{B}(a-4(1+d_1),n_1-a+1)\\
&=\frac{\Gamma(n_1+1)}{\Gamma(a)\Gamma(n_1-a+1)}\frac{\Gamma(a-4d)\Gamma(n_1-a+1)}{\Gamma(n_1-4(1+d_1)+1)}\\
&=\frac{\Gamma(n_1+1)}{\Gamma(n_1-4(1+d_1)+1)}\frac{\Gamma(a-4(1+d_1))}{\Gamma(a)}\\
&\lesssim n_1^{4(1+d_1)}a^{-4(1+d_1)}\\
&=\left(\frac{a}{n_1}\right)^{-4(1+d_1)},
\end{aligned}$$

where we used Lemma S.3.6 to obtain the inequality. Moreover,

$$\begin{aligned}
E\left[\left(\xi_{(b)}-\xi_{(a)}\right)^4\right]&=E\left[\mathrm{Beta}(b-a,n_1-b+a+1)^4\right]\\
&=\prod_{r=0}^{3}\frac{b-a+r}{n_1+d+r}\\
&\asymp\left(\frac{b-a}{n_1}\right)^4.
\end{aligned}$$

Combining with (S.3.2), we obtain

$$E\left[\tilde{\xi}^{-2(1+d_1)}(1-\tilde{\xi})^{-2(1+d_2)}\left(\xi_{(b)}-\xi_{(a)}\right)^2\mathbb{1}\left\{\tilde{\xi}\leq 1/2\right\}\right]\lesssim\left(\frac{a}{n_1}\right)^{-2(1+d_1)}\left(\frac{b-a}{n_1}\right)^2,$$

and similarly,

$$E\left[\tilde{\xi}^{-2(1+d_1)}(1-\tilde{\xi})^{-2(1+d_2)}\left(\xi_{(b)}-\xi_{(a)}\right)^2\mathbb{1}\left\{\tilde{\xi}\geq 1/2\right\}\right]\lesssim\left(\frac{n_1-b}{n_1}\right)^{-2(1+d_2)}\left(\frac{b-a}{n_1}\right)^2.$$

Therefore, we obtain

$$\begin{aligned}E\left[\left(Y_{(b)}-Y_{(a)}\right)^2\right]&\lesssim\left[\left(\frac{a}{n_1}\right)^{-2(1+d_1)}+\left(\frac{n_1-b}{n_1}\right)^{-2(1+d_2)}\right]\left(\frac{b-a}{n_1}\right)^2\\&\asymp\left[\left(\frac{a}{n_1}\right)^{-(1+d_1)}\left(\frac{n_1-b}{n_1}\right)^{-(1+d_2)}\left(\frac{b-a}{n_1}\right)\right]^2.\end{aligned}$$

This completes the case $a\geq 10$ and $b\leq n_1/2-10$.

2. Fix $k\leq 10$. We recall that, for any $u,v\in\mathbb{R}$, the joint density of $(Y_{(k)},Y_{(k+1)})$ is

$$f_{Y_{(k)},Y_{(k+1)}}(u,v)=f_Y(u)f_Y(v)F_Y(u)^{k-1}\bar{F}_Y(v)^{n_1-k-1}\frac{n_1!}{(k-1)!(n_1-k-1)!}\mathbb{1}\left\{u<v\right\}$$

(see, e.g. Casella and Berger, 2001, Theorem 5.4.6). We use the change of variables $s=F_Y(u)$ and $t=F_Y(v)$ and the notation $d\vec{x}=dx_1dx_2$. We obtain

$$\begin{aligned}&E\left[\left(Y_{(k+1)}-Y_{(k)}\right)^2\right]\\&=\int_{\mathbb{R}^2}(v-u)^2f_Y(u)f_Y(v)F_Y(u)^{k-1}\bar{F}_Y(v)^{n_1-k-1}\frac{n_1!\,\mathbb{1}\left\{u<v\right\}}{(k-1)!(n_1-k-1)!}dudv\\&=\int_{(0,1)^2}\left(F_Y^{-1}(t)-F_Y^{-1}(s)\right)^2s^{k-1}(1-t)^{n_1-k-1}\frac{n_1!\,\mathbb{1}\left\{s<t\right\}}{(k-1)!(n_1-k-1)!}dsdt\\&=\int_{(0,1)^2}\left(\int_s^t(F_Y^{-1})'(x)dx\right)^2s^{k-1}(1-t)^{n_1-k-1}\frac{n_1!\,\mathbb{1}\left\{s<t\right\}}{(k-1)!(n_1-k-1)!}dsdt\\&\leq\int_{(0,1)^2}\left(\int_s^t\left(x^{-(1+d_1)}(1-x)^{-(1+d_2)}\right)dx\right)^2s^{k-1}(1-t)^{n_1-k-1}\frac{n_1!\,\mathbb{1}\left\{s<t\right\}}{(k-1)!(n_1-k-1)!}dsdt\\&=\int_{(0,1)^2}\left(\prod_{i=1}^2\int_s^t\left(x_i^{-(1+d_1)}(1-x_i)^{-(1+d_2)}\right)dx_i\right)s^{k-1}(1-t)^{n_1-k-1}\frac{n_1!\,\mathbb{1}\left\{s<t\right\}}{(k-1)!(n_1-k-1)!}dsdt\\&=\int_{(0,1)^4}s^{k-1}\mathbb{1}\left\{s<\min_j x_j\right\}(1-t)^{n_1-k-1}\mathbb{1}\left\{t<\max_j x_j\right\}\\&\qquad\times\prod_{i=1}^2\left(x_i^{-(1+d_1)}(1-x_i)^{-(1+d_2)}\right)\frac{n_1!}{(k-1)!(n_1-k-1)!}ds\,dt\,d\vec{x}\\&=\int_{(0,1)^2}\frac{\min_j x_j^k}{k}\cdot\frac{(1-\max_j x_j)^{n_1-k}}{n_1-k}\prod_{i=1}^2\left(x_i^{-(1+d_1)}(1-x_i)^{-(1+d_2)}\right)\frac{n_1!}{(k-1)!(n_1-k-1)!}d\vec{x}\\&\leq\int_{(0,1)^2}\prod_{i=1}^2x_i^{k/2-(1+d_1)}(1-x_i)^{(n_1-k)/2-(1+d_2)}\binom{n_1}{k}d\vec{x}\\&=\binom{n_1}{k}\left(\int_0^1x^{k/2-(1+d_1)}(1-x)^{(n_1-k)/2-(1+d_2)}dx\right)^2\end{aligned}$$

$$
\begin{aligned}
&= \binom{n_1}{k} \left[\mathrm{B}\left(\frac{k}{2} - d_1, \frac{n_1 - k}{2} - d_2 \right) \right]^2 \\
&= \frac{\Gamma(n_1 + 1)}{\Gamma(k+1)\Gamma(n_1 - k + 1)} \left[\frac{\Gamma(\frac{k}{2} - d_1)\Gamma(\frac{n_1 - k}{2} - d_2)}{\Gamma(\frac{n_1}{2} - d_1 - d_2)} \right]^2 .
\end{aligned}
$$

Now, note that $\Gamma(\frac{k}{2} - d_1)$ is a constant, and

$$
\frac{\Gamma(n_1 + 1)}{\Gamma(k+1)\Gamma(n_1 - k + 1)} \left[\frac{\Gamma(\frac{k}{2} - d_1)\Gamma(\frac{n_1 - k}{2} - d_2)}{\Gamma(\frac{n_1}{2} - d_1 - d_2)} \right]^2 \asymp n_1^k \left(n_1^{d_1 - k/2} \right)^2 = n_1^{2d_1}.
$$

This concludes the proof. $\square$

Lemma S.3.6 *Let $\varepsilon > 0$. For all $a \in (-10, 10)$ and $x \in (0, \infty)$ such that $x + a > \varepsilon$, it holds that*

$$
\begin{aligned}
\frac{\Gamma(x+a)}{\Gamma(x)} &\leq (x + \lfloor a \rfloor)^a \qquad \textit{if } a > 0, \\
\frac{\Gamma(x+a)}{\Gamma(x)} &\leq \frac{(x - \lfloor |a| \rfloor)^a}{\left(1 + \frac{1}{1+\varepsilon}\right)^{\lfloor |a| \rfloor + 1}} \qquad \textit{if } a < 0.
\end{aligned}
$$

Proof: First, suppose that $0 < a < 1$. Then, for all $x > 0$,

$$
\frac{\Gamma(x+a)}{\Gamma(x)} \leq x^a \leq (x + \lfloor a \rfloor)^a,
$$

where the first inequality follows from the right part of Wendel (1948)'s double inequality. Second, suppose that $a \geq 1$. Let $u \in (0, 1)$ and define $b = a/u$. We have

$$
\frac{\Gamma(x+a)}{\Gamma(x)} = \frac{\Gamma(x+u)}{\Gamma(x)} \cdot \frac{\Gamma(x+2u)}{\Gamma(x+u)} \cdots \frac{\Gamma(x + \lfloor b \rfloor u)}{\Gamma(x + (\lfloor b \rfloor - 1)u)} \cdot \frac{\Gamma(x + bu)}{\Gamma(x + \lfloor b \rfloor u)}.
$$

By applying Wendel (1948)'s inequality to each term in the product, we obtain

$$
\begin{aligned}
\frac{\Gamma(x+a)}{\Gamma(x)} &\leq x^u (x+u)^u \cdots (x + (\lfloor b \rfloor - 1)u)^u \cdot (x + \lfloor b \rfloor u)^{u(b - \lfloor b \rfloor)} \\
&\leq (x + (\lfloor b \rfloor - 1)u)^{u \lfloor b \rfloor} (x + \lfloor b \rfloor u)^{u(b - \lfloor b \rfloor)} \\
&\leq (x + \lfloor b \rfloor u)^{u \lfloor b \rfloor} (x + \lfloor b \rfloor u)^{u(b - \lfloor b \rfloor)} \\
&= (x + \lfloor b \rfloor u)^a.
\end{aligned}
$$

Since $b \mapsto \lfloor b \rfloor$ is right-continuous, by taking the limit as $u \to 1$ we obtain

$$
\frac{\Gamma(x+a)}{\Gamma(x)} \leq (x + \lfloor a \rfloor)^a.
$$

Third, suppose that $-1 < a < 0$ and let $\tilde{x} = x + a$. Then,

$$
\frac{\Gamma(x+a)}{\Gamma(x)} = \frac{\Gamma(\tilde{x})}{\Gamma(\tilde{x} + (-a))} \leq \tilde{x}^a \left(\frac{\tilde{x} + (-a)}{\tilde{x}} \right)^{1-(-a)}
$$

$$
\begin{aligned}
&= x^a \cdot \frac{x}{x+a} \\
&\leq \frac{x^a}{1+\frac{a}{\varepsilon - a}} \\
&\leq \frac{x^a}{1-\frac{1}{1+\varepsilon}},
\end{aligned} \tag{S.3.3}
$$

where the first inequality follows from the left part of Wendel (1948)'s double inequality since $0 < (-a) < 1$ and $\tilde{x} > 0$, the second inequality follows from $x + a > \varepsilon$, and the third inequality follows from $a > -1$.

Fourth, suppose that $a \leq -1$ and let $u \in (-1, 0)$ such that $b = a/u$. We have

$$
\frac{\Gamma(x+a)}{\Gamma(x)} = \frac{\Gamma(x+u)}{\Gamma(x)} \cdot \frac{\Gamma(x+2u)}{\Gamma(x+u)} \cdots \frac{\Gamma(x+\lfloor b \rfloor u)}{\Gamma(x+(\lfloor b \rfloor - 1)u)} \cdot \frac{\Gamma(x+bu)}{\Gamma(x+\lfloor b \rfloor u)}.
$$

By applying inequality (S.3.3) to each term in the product, we obtain

$$
\begin{aligned}
\frac{\Gamma(x+a)}{\Gamma(x)} &\leq \frac{1}{\left(1-\frac{1}{1+\varepsilon}\right)^{\lfloor b \rfloor + 1}} x^u (x+u)^u \cdots (x+(\lfloor b \rfloor - 1)u)^u (x+\lfloor b \rfloor u)^{u(b-\lfloor b \rfloor)} \\
&\leq \frac{1}{\left(1-\frac{1}{1+\varepsilon}\right)^{\lfloor b \rfloor + 1}} (x+\lfloor b \rfloor u)^a.
\end{aligned}
$$

Since $b \mapsto \lfloor b \rfloor$ is right-continuous, by taking the limit as $u \to -1$ we obtain

$$
\frac{\Gamma(x+a)}{\Gamma(x)} \leq \frac{1}{\left(1-\frac{1}{\varepsilon+1}\right)^{\lfloor |a| \rfloor + 1}} (x - \lfloor |a| \rfloor)^a.
$$

□

Lemma S.3.7 *Let $s, t \in [0,1]$ and assume that $|s - t| \leq \varepsilon(s\bar{s} + t\bar{t})$ for some $\varepsilon \leq 1/2$. Then $t\bar{t} \leq 6\, s\bar{s}$, and $s\bar{s} \leq 6\, t\bar{t}$.*

Proof: The second claim follows from the first one by symmetry of the roles of s and t. We therefore focus on proving the first claim.

Without loss of generality, assume $s \leq 1/2$ (otherwise, it suffices to repeat the following argument with $\bar{s}$ and $\bar{t}$ instead of s and t, respectively). If $t \leq s$, then $t\bar{t} \leq s\bar{s}$ and the result follows. Assume from now on that $t > s$. By assumption, we have

$$
t - s \leq \varepsilon(s\bar{s} + t\bar{t}) \leq \varepsilon(s + t) \qquad \text{i.e.} \qquad t \leq \frac{1+\varepsilon}{1-\varepsilon} s,
$$

by rearranging the terms. Hence

$$
t\bar{t} \leq t \leq \frac{1+\varepsilon}{1-\varepsilon} s \leq \frac{1+\varepsilon}{1-\varepsilon} s \cdot 2\bar{s} \leq 6 s\bar{s}.
$$

This completes the proof. □

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